Beyond True and False: Logic, Algebra and Topology

Florence 3 - 5 December 2014

Schedule and Abstracts

Foreword

In this booklet we included the schedule and the abstracts of the talks presented at Beyond 2014 (*Beyond True and False: Logic, Algebra and Topology*), held at the University of Florence (3-5 December 2014). The workshop brought together experts in many valued logic, proof theory and algebraic logic, with the goal of cross-fertilisation.

This meeting will also include the presentation of the new ERCIM Working Group on Many-Valued Logics (ManyVal, https://wiki.ercim.eu/wg/ManyVal/index.php/Main_Page).

We would like to thank our sponsors: the Department of Computer Science, Statistics and Applications 'G. Parenti' and the EU FP7 programme (IEF-Grant Agreement n. 326202).

Leonardo M. Cabrer

on behalf of the Programme Committee:
Vincenzo Marra
Daniele Mundici
Maria C. Verri

Florence, 3 December 2014

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Schedule

Wednesday, December 3

8.00–8.50		Registration(1)
8.50–9.00		Welcome(2)
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9.35–10.10	S. van Gool	: Duality for sheaf representations of distributive lattices
10.10–10.55		Coffee break(3)
10.55–11.30	I. Leuştean	: Applications of the semisimple tensor product of MV-algebras
11.30–12.05	F. Marigo	: Finite GBL-algebras and Heyting algebras with equivalence relations
12.05–12.40	M. Busaniche	: Representation of BL-algebras with finite and independent spectrum
12.40–14.20		Lunch break(4)
14.20–14.55	V. Marra	: Γ
14.55–15.30	L. Reggio	: Stone duality above dimension zero: Infinitary algebras of real-valued functions on compact Hausdorff spaces
15.30–16.15		Coffee break
16.15–16.50	B. Gerla	: Categorical equivalences of classes of MTL-algebras through cancellative hoops
16.50–17.10	P. Cintula	: Presentation of the ERCIM working group on Many-Valued Logics (ManyVal)

Thursday, December 4

9.00–9.35	F. Montagna	: Variants of Ulam game and game semantics for many-valued logics
9.35–10.10	T. Kroupa	: Balanced McNaughton Functions
10.10–10.55		coffee break
10.55–11.30	A. Palmigiano	: Unified Correspondence as a Proof-Theoretic Tool
11.30–12.05	P. Oliva	: Hoops and Intuitionistic Łukasiewicz Logic
12.05–12.40	A. Ciabattoni	: A proof theoretic approach to standard completeness
12.40–14.20		Lunch break
14.20–14.55	F. Paoli	: Projectable ℓ-groups and algebras of logic
14.55–15.30	R. Horčík	: Full Lambek Calculus with Contraction is Undecidable
15.30–16.15		Coffee break
16.15–16.50	P. Cintula	: Combining logics using two-layer modal syntax
16.50–17.25	A. Kurz	: On a category theoretic perspective of many-valued logic
20.00–22.00		Social Dinner(5)

Friday, December 5

9.00–9.35	O. Caramello	: The Morita-equivalence between MV-algebras and abelian ℓ-groups with strong unit
9.35–10.10	A.C. Russo	: Lattice-ordered abelian groups and perfect MV-algebras: a topos-theoretic perspective
10.10–10.55		Coffee break
10.55–11.30	D. Diaconescu	: Mutually exclusive nuances of truth
11.30–12.05	S. Lapenta	: Baker-Beynon duality for Riesz MV-algebras
12.05–12.40	S. Aguzzoli	: Towards a Riesz representation theorem for finite Heyting algebras
12.40–14.20		Lunch break
14.20–14.55	A. Pedrini	: From Freudental's Spectral Theorem to projectable hulls of unital Archimedean lattice-groups, through compactification of minimal spectra. Part 1: the compact case.
14.55–15.30	D. McNeill	: Part 2: the general case.
15.30–16.15		Coffee break
16.15–16.50	L. Spada	: A(nother) duality for the whole variety of MV-algebras
16.50–17.25	G. Metcalfe	: Exact Unification and Admissibility

(1) The registration will take place on the first floor of the Dipartimento di Statistica, Informatica, Applicazioni "G. Parenti" (DISIA), vialle Morgagni 59.

(2) All the talks will be delivered in Room 32 third floor of the DISIA.

(3) During the coffee breaks we will offer some refreshment in the Tea Room in the first floor the DISIA.

(4) All the speakers will be provided with lunch vouchers at the registration desk on the first day.

(5) The social dinner is at "Ristorante Cafaggi", via Guelfa 35 Rosso. Tel: +39 055 294989 www.ristorantecafaggi.com

Towards a Riesz representation theorem for finite Heyting algebras

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Joint work in progress with V. Marra.

Representation of BL-algebras with finite and independent spectrum

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Joint work with S. Aguzzoli, J. L. Castiglioni and N. Lubomirsky.

Representation theorems for classes of algebras are one of the most important tools to work within the class. They allow to understand the elements of the class in terms of simpler or better known structures, and also to compare different algebras.

For the case of the variety of BL-algebras, the algebraic counterpart of basic logic ([4]), there are representation theorems for the class of finite algebras [3] and for the class of totally ordered algebras (BL-chains, [1]).

As an attempt to extend the representation for BL-chains to the general case, the notion of poset sum is introduced in [5] (called poset product in [2]). With this construction an embedding theorem can be proved, which states that each BL-algebra can be embedded into the poset product of MV-chains and product chains, two particular cases of BL-chains. So each BL-algebra can be seen as a subalgebra of a poset product.

In this talk a representation theorem for a class of BL-algebras will be presented. These are algebras with finite and independent prime spectrum. Such class properly includes the class of finite algebras. The idea of independent spectrum will be explained during the talk.

The representation relies on the decomposition theorem for BL-chains that states that each nontrivial BL-chain can be uniquely decomposed as an ordinal sum of nontrivial totally ordered Wajsberg hoops. We will show that a BL-algebra $\mathbf{A}$ with finite and independent spectrum P is isomorphic to an algebra of functions $R(\mathbf{A})$. For each $p \in P$ we define a totally ordered Wajsberg hoop $\mathbf{W}_p$. The domain of each function f of $R(\mathbf{A})$ is a subset $P_f \subseteq P$ and $f(p) \in \mathbf{W}_p$.

The proof of the representation takes into account the embedding of each BL-algebra into a poset product.

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SEVENTH FRAMEWORK PROGRAMME MARIE CURIE ACTIONS

The Morita-equivalence between MV-algebras and abelian ℓ-groups with strong unit

Olivia Caramello

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This talk is based on [2]. We show that the theory $\mathbb{MV}$ of MV-algebras is Morita-equivalent to the theory $\mathbb{L}_u$ of abelian ℓ-groups with strong unit. This generalizes the well-known equivalence between the categories of set-based models of these two theories established by D. Mundici in [3] and allows to apply the 'bridge technique' of [1] to transfer properties and results from one theory to the other, obtaining new insights which are not visible by using classical techniques. Among these results, we mention a bijective correspondence between the geometric theory extensions of the theory $\mathbb{MV}$ and those of the theory $\mathbb{L}_u$, a form of completeness and compactness for the infinitary theory $\mathbb{L}_u$, a logical characterization of the finitely presentable ℓ-groups with strong unit and a sheaf-theoretic version of Mundici's equivalence.

References

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A proof theoretic approach to standard completeness

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Joint work with P. Baldi.

Standard completeness, that is completeness of a logic with respect to algebras based on truth values in $[0, 1]$ has received increasing attention in the last years. In a standard complete logic connectives are interpreted by suitable functions on $[0, 1]$, and this makes it a fuzzy logic in the sense of Hájek.

In this talk we present a proof of standard completeness that uniformly applies to many axiomatic extensions of Uninorm Logic UL. Its core is a general proof of the elimination of the Takeuti and Titani density rule from derivations in Gentzen-style (hypersequent) calculi. Our proof applies to all logics lying between UL and MTL already known to be standard complete and allows for the discovery of new uninorm-based fuzzy logics. The latter include UL extended with contraction or mingle.

Combining logics using two-layer modal syntax

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Joint work with Carles Noguera

Two-layer modal syntax is based on two propositional languages (lower and upper one) and a modal language (a collection of modalities together with their arities) and features three kinds of formulae: (i) non-modal formulae build using *lower* propositional language, (ii) atomic modal formulae obtained by applying the modalities to *non-modal* ones, and (iii) complex modal formulae built from the atomic ones using the *upper* propositional language (i.e., the modalities cannot be nested and propositional languages cannot be mixed).

Early examples of logics employing this kind of syntax were modal logics of uncertainty inspired by Hamblin's seminal idea of reading the modal operator $P\varphi$ as 'probably φ' [6], meaning that the probability of φ is bigger than a given threshold (later elaborated by Fagin, Halpern and many others (see e.g. [2,5]).

These initial examples used the classical logic to govern the behavior of formulae of both modal and non-modal layer. An interesting departure from the classical paradigm has been proposed by Hájek and Harmancová in [4] (later developed in Hájek's monograph [3]): they kept the classical logic as interpretation of the lower syntactical layer, but proposed to use Łukasiewicz logic in the upper layer, so that the truth degree of $P\varphi$ could be directly identified with the probability of φ. Later, other authors changed even the logic governing in the lower layer (e.g., to other fuzzy logic in order to allow for a treatment of uncertainty of vague events).

This research gave rise to an interesting way of combining logics which allows to use one logic to reason about formulae (rules) of other logic. The aim of this talk is to propose foundations for further research in this promising area. We define an abstract notion of a two-layer syntax and logic, a general semantics of *measured* Kripke frames and prove two forms of completeness theorem.

Acknowledgements: This research is supported by grant GAP202/10/1826 of the Czech Science Foundation. Petr Cintula also acknowledges the support of RVO 67985807; Carles Noguera was also supported by the FP7 PIRSES-GA-2009-247584 project MaToMUVI. This talk is based on [1].

References

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SEVENTH FRAMEWORK PROGRAMME
MARIE CURIE ACTIONS

Mutually exclusive nuances of truth

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Joint work with I. Leuştean.

Nuances of truth represent a robust paradigm in the framework of many-valued logics [1]. The idea of nuancing states that a many-valued object is uniquely determined by some Boolean objects, its nuances, and it is called the determination principle. However, a many-valued object cannot be recovered only from its Boolean nuances. This idea goes back to Gr. C. Moisil [4] and it is mathematically expressed by a categorical adjunction between Boolean algebras and Łukasiewicz-Moisil algebras. Moisil's determination principle cannot be extended for subalgebras: distinct Łukasiewicz-Moisil algebras can have the same Boolean algebra reduct.

In this talk we explore a more expressible notion of nuances, namely mutually exclusive nuances of truth (or disjoint nuances of truth, for short). This idea was started in [3] and continued in [2]. Mutually exclusive nuances of truth, apart from saving the determination principle for subalgebras, give a new perspective on how Stone-type duality can be obtained for Łukasiewicz-Moisil algebras starting from Stone spaces.

References

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Ultrafilter equations

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A powerful tool for characterizing classes of regular languages is provided by Reitermans theory of profinite equations. Stone duality allows a generalization of this theory to so-called ultrafilter equations which are available for complexity classes beyond the regular fragment. In this talk we give an introduction to this concept and show how it may be used to characterize a certain class of languages related to a Boolean circuit complexity class.

Categorical equivalences of classes of MTL-algebras through cancellative hoops

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Joint work with S. Aguzzoli

Hoops have been widely used in relation with algebraic structures of many-valued logics, mainly in the framework of MTL-algebras. In this work we focus on cancellative hoops and we use them to build up some classes of MTL-algebras, giving a common framework to old and new results.

Indeed, starting from a cancellative hoop we can make three kind of operations that lead to the construction of different algebraic structures (see [2], [3], [4], [5]):

- If we add a bottom element to a cancellative hoop we obtain a product algebra and every directly indecomposable product algebra is of this kind.
- If we make a disconnected rotation of a cancellative hoop we have an MV-algebra belonging to the variety $V(C)$ generated by the Chang MV-algebra, and again all directly indecomposable MV-algebras in this variety are the disconnected rotation of a cancellative hoop.
- Finally, if we make the connected rotation of a cancellative hoop we obtain an IMTL-algebra in the variety $\mathbb{J}\Pi$ generated by the standard IMTL-algebra given by the Jenei rotation of the product t-norm, and every directly indecomposable IMTL-algebra in this variety is either a connected or a disconnected rotation of a cancellative hoop.

Using such results on directly irreducible algebraic structures, we can state a categorical equivalence between the category of product algebras and the category of MV-algebras belonging to the variety $V(C)$. Further we can consider the category Δ whose objects are pairs (A, F) made by an MV-algebra A in $V(C)$ and a filter of the Boolean skeleton $B(A)$ of A, and which arrows $f : (A, F) \to (A', F')$ are made by an MV-homomorphism $f : A \to B$ such that $f(F) \subseteq F'$. Using the characterization of free objects in the corresponding varieties, we can hence show that the category of finitely generated $\mathbb{J}\Pi$-algebras is equivalent to the full subcategory of Δ in which the objects are made by a finitely generated MV-algebra and a principal filter of the Boolean skeleton. Starting from this, taking filtered colimits in the corresponding categories, we show that the category of $\mathbb{J}\Pi$-algebras is equivalent to the category Δ.

The variety $\mathbb{J}\Pi$ has deep relations with MV-algebras: indeed MV-algebras in $\mathbb{J}\Pi$ form a variety that coincides with the variety generated by the three element MV-chain and by Chang's MV-algebra. As it is shown in [1], this allows to prove that the variety of MV-algebras cannot be axiomatised by one variable axioms starting from IMTL and hence, also the axiomatisation of MV from MTL and from BL cannot use only one variable From another perspective, $\mathbb{J}\Pi$ can be seen as the smallest standard generated variety of MTL-algebras containing the variety $V(C)$.

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SEVENTH FRAMEWORK PROGRAMME
MARIE CURIE ACTIONS

Full Lambek Calculus with Contraction is Undecidable

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Joint work with K. Chvalovský

Besides the cut rule, Gentzen's sequent calculus **LJ** for propositional intuitionistic logic contains other structural rules, namely the rule of contraction (c), exchange (e), left weakening (i) and right weakening (o). By removing all these rules from **LJ**, one arrives at the full Lambek calculus **FL**. More generally, every extension of **FL** by a subset of the rules (c), (e), (i) and (o) defines a logic between **FL** and **LJ**. In [1] these logics are called *basic substructural logics*. It is known that each of these logics has an analytic sequent calculus. In particular, the cut rule is eliminable in all these calculi if the contraction rule is introduced in its global variant (for survey see e.g. [1, Chapter 4]).

Cut elimination is closely related to decidability. It is known that all basic substructural logics are decidable except of $\mathbf{FL_c}$ and $\mathbf{FL_{co}}$ where the former is the extension of **FL** by the contraction rule and the latter is the extension of $\mathbf{FL_c}$ by the right weakening rule. The decidability of basic substructural logics without the contraction rule follows immediately from the cut elimination theorem and is proved in [5]. On the other hand, such an easy argument is not applicable for logics with the contraction rule since this rule makes the proof-search tree infinite. Nevertheless, intuitionistic logic is decidable [2,3] and the same holds for the extension of **FL** by the exchange and contraction rule [4] (the original combinatorial idea from the proof goes back to Kripke [6]). In contrast, we show that $\mathbf{FL_c}$ and $\mathbf{FL_{co}}$ are the only undecidable logics among all basic substructural logics. In fact, we prove that their common positive fragment $\mathbf{FL_c^+}$ is undecidable.

References

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Balanced McNaughton Functions

Tomáš Kroupa

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We will present a problem motivated by the construction of robust and balanced Boolean function by Linial and Ben-Or. The goal is to describe the class of McNaughton functions/$\mathbb{Z}$-maps that transform uniform random inputs to a uniform random output. The special cases of such maps are $\mathbb{Z}$-homeomorphisms of the unit cube, which preserve the Lebesgue measure. In the talk we will formulate a necessary and sufficient condition for balancedness of McNaughton function (found by Panti) and show some refinements to this theorem.

Acknowledgments: The author gratefully acknowledges the support from Marie Curie Intra-European Fellowship OASIG (PIEF-GA-2013-622645).

On a category theoretic perspective of many-valued logic

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In a well-known paper from 1973 on metric spaces and generalized logic (available as TAC reprint No.1), Lawvere showed how categories enriched over a lattice (or quantale) embody a many-valued logic. Taking this as a starting point, we will show how MV-algebras arise from this perspective.

Baker-Beynon duality for Riesz MV-algebras

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Joint work with A. Di Nola and I. Leuştean

The Baker-Beynon duality [1, 2] is a duality between lattice-ordered structures and suitable subspaces of $\mathbb{R}^n$, for some n.

We will focus our attention on the case of Riesz Spaces with strong unit, and consequently on Riesz MV-algebras. Therefore, we present work in progress on the Baker-Beynon duality for Riesz MV-algebras. We recall that a Riesz MV-algebra is an MV-algebra endowed with a scalar product with scalars from $[0, 1]$.
Denoted by **fgpRMV** the category of finitely generated and projective Riesz MV-algebras with homomorphisms of Riesz MV-algebras and $\mathcal{P}_{\mathbb{R}}$ the category of polyhedra with piecewise linear maps, our first result is the following.

Theorem 1 *The category* **fgpRMV** *is equivalent to the dual of the category* $\mathcal{P}_{\mathbb{R}}$.

Similar results hold for MV-algebras [5, 4], where the structures involved are finitely presented MV-algebras.
We connect finitely presented MV-algebras, finitely generated MV-algebras and projective MV-algebras to finitely presented Riesz MV-algebras, finitely generated Riesz MV-algebras and projective Riesz MV-algebras respectively. The main tool is the semisimple tensor product of MV-algebra: in [3] we have displayed an adjunction between semisimple MV-algebras and semisimple Riesz MV-algebras. From MV-algebras to Riesz MV-algebras we get the following.

Theorem 2 *i) Let A be a semisimple MV-algebra. If A is finitely generated, then $[0, 1] \otimes A$ is a finitely generated Riesz MV-algebra;*

ii) Let A be a semisimple MV-algebra. If A is projective, then $[0, 1] \otimes A$ is a projective Riesz MV-algebra;

iii) Let A be a semisimple MV-algebra. If A is finitely presented, then $[0, 1] \otimes A$ is a projective Riesz MV-algebra.

On the other side, if $\mathcal{U}_{\mathbb{R}}$ is the usual forgetful functor from Riesz MV-algebras to MV-algebras, we have the following.

Theorem 3 *i) Let A be a Riesz MV-algebra. If $\mathcal{U}_{\mathbb{R}}(A)$ is projective, then A is projective;*

ii) Let R be a Riesz MV-algebra. If $\mathcal{U}_{\mathbb{R}}(R)$ is finitely generated, then R is finitely generated.

Finally, in order to connect finitely presented structures, we deal with two special categories.

i) MV_{fp}^{pres} is the category whose objects are couple (X, I) with X non-empty set and I principal ideal in the free MV-algebra generated by X and usual homomorphism of MV-algebras;

ii) RMV_{fp}^{pres} is the category whose objects are couple (X, I) with X non-empty set and I principal ideal in the free Riesz MV-algebra generated by X and usual homomorphism of Riesz MV-algebras.
We prove display an equivalence between MV_{fp}^{pres} and RMV_{fp}^{pres}. Moreover, any object of MV_{fp}^{pres}

(RMV_{fp}^{pres} respectively) is in one-one correspondence with a suitable finitely presented MV-algebra (Riesz MV-algebra) and any morphism in MV_{fp}^{pres} or RMV_{fp}^{pres} induce a morphism between finitely presented MV-algebras or Riesz MV-algebras.

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Applications of the semisimple tensor product of MV-algebras

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Joint work with S. Lapenta.

Since the real interval $[0,1]$ is closed to the product operation, a natural problem was to find a complete axiomatization for the variety generated by the standard MV-algebra $([0,1], \oplus, \neg, 0)$ endowed with the real product. If the product operation is defined as a bilinear function $\cdot : [0,1] \times [0,1] \to [0,1]$ then the standard model is $([0,1], \oplus, \cdot, \neg, 0)$; if the product is a scalar multiplication then the standard model is $([0,1], \oplus, \{r \mid r \in [0,1]\}, \neg, 0)$, where the function $x \mapsto rx$ is a linear for any $r \in [0,1]$. The approach based on the internal binary product led to the notion of *PMV-algebra* [1,6,7], while the approach based on the scalar multiplication led to the notion of *Riesz MV-algebra* [2]. In [4] we defined the *fMV-algebras* adding both an internal product and a scalar multiplication; in this case, our standard model is $([0,1], \oplus, \cdot, \{r \mid r \in [0,1]\}, \neg, 0)$. The varieties of MV-algebras and Riesz MV-algebras are generated by their corresponding standard models; this is not the case for PMV-algebras and fMV-algebras: in these classes the standard models generate proper sub-varieties.

There are obvious forgetful functors between the above mentioned classes of structures:

$$\begin{tikzcd}
\mathbf{MV} & \mathbf{PMV} \arrow[l, "U"'] \\
\mathbf{RMV} \arrow[u, "U"] & \mathbf{fMV} \arrow[l, "U"] \arrow[u, "U"'] \arrow[ul, "U"']
\end{tikzcd}$$

Our goal is to define the left adjoint functors. We do this in [3] for the corresponding subclasses of semisimple structures using the semisimple MV-algebraic tensor product defined in [8] and its scalar extension property [5]. Note that all algebras with product are considered unital. Of particular importance is the *semisimple tensor PMV-algebra of an MV-algebra*, whose construction is inspired by the well-known definition of the tensor algebra. The main result asserts that the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{MV_{ss}} \arrow[r, bend left, "\mathbf{T}"] \arrow[d, bend right, "\mathcal{T}_{\otimes}"'] & \mathbf{PMV_{ss}} \arrow[l, "U"] \arrow[d, bend left, "\mathcal{F}_{\otimes}"] \\
\mathbf{RMV_{ss}} \arrow[u, "U"] \arrow[r, bend right, "\mathcal{F}_{\mathbf{T}}"'] & \mathbf{fMV_{ss}} \arrow[l, "U"'] \arrow[u, "U"] \arrow[ul, "U"']
\end{tikzcd}$$

As consequence we prove the amalgamation property for semisimple PMV-algebras, semisimple Riesz MV-algebras and semisimple fMV-algebras.

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Finite GBL-algebras and Heyting algebras with equivalence relations

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Joint work with T. Flaminio and B. Gerla

Our work attempts to describe models of fuzzy logics making use of crisp logics and indiscernibility relations.

We want to generalize the work that, in the algebraic context, was done for instance in [2], where some classes of MV-algebras are represented by Boolean algebras equipped with a distinguished automorphism, in [3], where all MV-algebras are represented by Boolean algebras with an equivalence relation given by a subgroup of the automorphism group, and in [5], where a similar representation is given for MV-algebras and BL-algebras.

We follow the approach of [3], but we take Heyting algebras in place of Boolean algebras, in the spirit of [5]. Namely, we consider a Heyting algebra, counterpart of an intuitionistic propositional theory, and we define an *indiscernibility relation* on it. We make two hypothesis on this relation:

- it is an equivalence relation given by a subgroup G of the automorphism group of the Heyting algebra: two elements are indiscernible if there is an automorphism in G mapping one to the other;

- if two chains are element-wise indiscernible, then there is an unique automorphism in G bringing one chain to the other.

In analogy with the BG-pairs mentioned in [3], we call *HG-pair* the pair made of a Heyting algebra and an indiscernibility relation. We describe the category of finite HG-pairs and we show that it is dually equivalent to a category WP^* of weighted partially ordered sets and open maps respecting weights in a certain sense.

Finite HG-pairs are also in correspondence with finite *GBL-algebras*. Commutative, integral GBL-algebras are defined as commutative, integral, divisible residuated lattices, or equivalently as hoops with a lattice reduct. The variety of commutative, integral GBL-algebras extends the variety of MV-algebras and the one of Heyting algebras and it can be viewed as a fuzzy generalization of the latter. Informally, GBL-algebras generalize Heyting algebras in a similar way as MV-algebras generalize Boolean algebras.

Finite GBL-algebras are always commutative and, as shown in [4], they can be represented as poset products of Wajsberg chains. This result brings a natural correspondence between the category of finite GBL-algebras with homomorphisms and a category of weighted posets whose class of morphisms extends the one of WP^*. A similar representation for the subclass of finite BL-algebras is given in [1].

We retrace the duality of finite GBL-algebras and weighted posets adding that little bit of structure needed to recover a category of algebras equivalent to the category of finite HG-pairs. If we add to finite GBL-algebras a further binary operation $\oplus$, which behaves like the sum in the subclass of MV-algebras and like the disjunction in the subclass of Heyting algebras, we have a category of algebras dual to WP^*. Thus we have an equivalence between the category of what we call *finite GBL$_\oplus$-algebras* and the category of finite HG-pairs.

Finite $\mathrm{GBL}_{\oplus}$-algebras are a *wide* subcategory of finite GBL-algebras, having the same class of objects and a subclass of morphisms. Indeed, if we define the lower and upper approximation of an element by idempotents, morphisms of GBL-algebras preserve only the lower approximation, while morphisms of $\mathrm{GBL}_{\oplus}$-algebras preserve both.

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Γ

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This talk aims at providing context and background for the results presented in Luca Reggio's talk on the axiomatisation of an infinitary variety dual to the category of compact Hausdorff spaces and their continuous maps. The key rôle of the mathematical construct referred to in the title is emphasised.

From Freudental's Spectral Theorem to projectable hulls of unital Archimedean lattice-groups, through compactification of minimal spectra part 2: the general case.

Daniel McNeill

Dipartimento di Scienze Teoriche e Applicate
University of Insubria

Joint work with R. N. Ball, V. Marra, and A. Pedrini.

This is the second part of a series of two presentations, the first being by Andrea Pedrini.

In its basic version, Freudenthal's Spectral Theorem [4] asserts that any element of a Riesz space R with a strong unit u and the principal projection property may be uniformly approximated, in the norm that u induces on R, by abstract characteristic functions – "components of the unit u". Freudenthal's theorem led to a considerable amount of research on Riesz spaces and their generalisations, the lattice-ordered Abelian groups that concern us here. (See [8] and [3, 6] for background.) One main line of research concentrated on extending one given structure G to a minimal completion that enjoys the principal projection property, where Freudenthal's theorem therefore applies. Such an extension is called *the projectable hull* of G. Here we present a new construction of the projectable hull of an Archimedean ℓ-group equipped with a strong order unit u that does not use direct limits, nor essential closures. Our construction exposes instead the intimate connection between projectable hulls and zero-dimensional compactifications of spectral spaces of minimal prime ideals.

In this second talk, we quickly review some of the basic definitions concerning the notions involved in the construction and particularly consider when the space $\operatorname{Min} G$ — a Hausdorff zero-dimensional space — is not compact [1]. For our construction in this case, we must recall the definition of a Wallman base for a topology as well as a Wallman compactification [9].

Here, the base $\{\mathbb{V}_m(g)\}_{g\in G}$ for the closed sets of $\operatorname{Min} G$ is *n*ot Boolean algebra under set-theoretic union, intersection and complementation and does *n*ot coincide with the set of all clopen subsets of $\operatorname{Min} G$ [2, 10].

However, we note that $\{\mathbb{V}_m(g)\}_{g\in G}$ does form a Wallman base for $\operatorname{Min} G$, consisting of clopen sets, from which we may construct a special zero-dimensional Wallman compactification of $\operatorname{Min} G$ which we denote $\beta_0^G(\operatorname{Min} G)$. Similar to the previous talk, we now have an embedding of G into $C(\beta_0^G(\operatorname{Min} G))$ and letting $\widetilde{G}$ be the image of G and K collection of continuous characteristic functions on $\beta_0^G(\operatorname{Min} G)$, we may construct the projectable hull of G, $\mathcal{P}(G)$, as the ℓ-subgroup of $\mathrm{C}(\beta_0^G(\operatorname{Min} G))$ generated by $\widetilde{G} \cup K$. We note that the compactification $\beta_0^G(\operatorname{Min} G)$ is specifically constructed to have only those continuous characteristic functions derived from G and that $\operatorname{Max} \mathcal{P}(G) \cong \beta_0^G(\operatorname{Min} G)$.

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SEVENTH FRAMEWORK PROGRAMME
MARIE CURIE ACTIONS

Exact Unification and Admissibility

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Joint work with L. Cabrer

In this talk I will describe a new hierarchy of "exact" unification types, motivated by the study of admissible rules, where unifiers of identities in an equational class are preordered, not by instantiation, but rather by inclusion over the corresponding sets of unified identities. Minimal complete sets of unifiers under this new preordering always have a smaller or equal cardinality than those provided by the standard instantiation preordering, and I will give examples – distributive lattices, idempotent semigroups, and MV-algebras – where a dramatic improvement occurs. I will also explain the algebraic interpretation of exact unification, inspired by Ghilardi's algebraic approach to equational unification.

Variants of Ulam game and game semantics for many-valued logics

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Joint work with E. Corsi.

As shown by Mundici [Mu1], the Rényi-Ulam game (that is, the Ulam game with a maximum number of lies) is a very interesting game, because it has applications to the treatment of uncertainty and at the same time it constitutes a sound and complete game semantics for Łukasiewicz logic. The idea is that in the Ulam game with $\leq e$ lies any sequence σ of questions-answers is coded by a truth function f_σ from the search space S into $\left\{0, \frac{1}{e+1}, \frac{2}{e+1}, \ldots, \frac{e}{e+1}, 1\right\}$, and the juxtaposition of two sequences σ and τ is coded by the Łukasiewicz conjunction of f_σ and f_τ. Moreover it is possible to represent the other connectives of Łukasiewicz logic in terms of operations on truth functions.

In [CMBL] the authors investigate a multichannel variant of the Ulam game, and prove that this variant constitutes a complete game semantics for Hàjek's logic BL. This variant admits the Rényi Ulam game as a special case. However, although both Gödel Logic and Product Logic extend BL, the game proposed by Cicalese and Mundici does not provide a game semantics for these logics. The goal of this talk is to present a complete game semantics for BL which includes a complete game semantics for Łukasiewicz, Gödel and Product Logics as special cases.

We start from the case of Gödel Logic. In this case lies are not admitted, but the answers can use different channels, and each channel has a fixed probability to work correctly. Moreover if a channel C works correctly, then all the answers using channel C will be correct, otherwise the answers may contain arbitrarily many lies.

In the case of Product Logic, we combine a probabilistic variant introduced by Pelc [P], in which Responder may lie with a fixed probability, and a variant by Cicalese and Mundici [CM], in which any answer YES must be correct, while an answer NO might be incorrect.

Finally, we consider a variant of the game semantics proposed in [CMBL], which not only constitutes a complete game semantics for BL, but avoids the requirement on increasingly noisy channels and admits semantics for Gödel and Product Logics as special cases.

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Hoops and Intuitionistic Łukasiewicz Logic

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Joint work with R. Arthan.

We study multi-valued logic of Łukasiewicz LL as an extension of intuitionistic affine logic IAL. Two extra axioms need to be added to IAL to obtain LL: the classical axiom of double negation elimination DNE, and the intuitionistically valid axiom of divisibility AD. The intermediate logic IAL + AD we call intuitionistically Łukasiewicz logic ILL. Hoops are the algebraic structures that provide natural models for IAL + AD. The logic ILL turns out to be surprisingly powerful fragments of intuitionistic logic, as we demonstrate by presenting:

- a simpler proof of the Ferreirim-Veroff-Spinks theorem;
- a proof that idempotent elements of a hoop form a sub-hoop;
- a proof that double negation is a hoop homomorphism; and
- proofs for intuitionistically valid versions of all the De Morgan dualities.

However most of the proofs are very intricate and have been found with computer assistance. We have factored these proofs (also with some computer assistance) into what we believe is a natural and understandable sequence of lemmas and theorems. The presentation is further simplified by the introduction of four derived connectives, which also satisfy natural De Morgan dualities. We conclude by showing that the homomorphism property of the double-negation mapping implies that all the standard negative translations of classical into intuitionistic Łukasiewicz coincide, as they do in full intuitionistic logic. This is in contrast with affine logic for which we will see that both Gentzen and Glivenko translations fail.

Unified Correspondence as a Proof-Theoretic Tool

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Delft University of Technology

Joint work with G. Greco, M. Ma, A. Tzimoulis and Z. Zhao.

This talk focuses on the formal connections which have recently been highlighted between correspondence phenomena, well known from the area of modal logic, and the theory of display calculi, originated by Belnap.

These connections have been seminally observed and exploited by Marcus Kracht, in the context of his characterisation of the modal axioms (which he calls primitive formulas) which can be effectively transformed into 'good' structural rules of display calculi. In this context, a rule is 'good' if adding it to a display calculus preserves Belnap's cut-elimination theorem.

In recent years, correspondence theory has been uniformly extended from classical modal logic to diverse families of nonclassical logics, ranging from (bi-)intuitionistic (modal) logics, linear, relevant and other substructural logics, to hybrid logics and mu-calculi. This generalisation has given rise to a theory called unified correspondence, the most important technical tool of which is the algorithm ALBA.

We put ALBA to work to obtain a generalisation of Kracht's transformation procedure from axioms into 'good' rules. This generalisation concerns more than one aspect. Firstly, we define primitive formulas/inequalities in any logic the algebraic semantics of which is based on distributive lattices with operators. Secondly, in the context of each such logic, we significantly generalise the class of primitive formulas/inequalities, and we apply ALBA to obtain an effective transformation procedure for each member of this class.

Time permitting, we will discuss the connections between the ALBA-aided transformation procedure and other similar procedures existing in the literature, developed for instance by Negri, Ciabattoni and other authors.

Projectable ℓ-groups and algebras of logic

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Joint work with J. Gil Férez, A. Ledda, and C. Tsinakis

P.F. Conrad and other authors launched a general program for the investigation of lattice-ordered groups, aimed at elucidating some order-theoretic properties of these algebras by inquiring into the structure of their lattices of convex ℓ-subgroups. Among its achievements, we mention a characterization of projectable ℓ-groups and their negative cones in terms of their order structure.

This approach can be naturally extended to residuated lattices and their convex subalgebras. In this broader perspective, we revisit the Galatos-Tsinakis categorical equivalence between integral generalized MV-algebras and negative cones of ℓ-groups with a nucleus, showing that it restricts to an equivalence of the full subcategories whose objects are the projectable members of these classes. Upon recalling that projectable integral generalized MV-algebras and negative cones of projectable ℓ-groups can be endowed with a positive Gödel implication, and turned into varieties by including this implication in their signature, we prove that there is an adjunction between the categories whose objects are the members of these varieties and whose morphisms are required to preserve implications.

From Freudental’s Spectral Theorem to projectable hulls of unital Archimedean lattice-groups, through compactification of minimal spectra Part 1: the compact case.

Andrea Pedrini

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Joint work with R. N. Ball, V. Marra, and D. McNeill.

This is the first part of a series of two abstracts, the second one being by Daniel McNeill.

In its basic version, Freudenthal’s Spectral Theorem [4] asserts that any element of a Riesz space R with a strong unit u and the principal projection property may be uniformly approximated, in the norm that u induces on R, by abstract characteristic functions – “components of the unit u”. Freudenthal’s theorem led to a considerable amount of research on Riesz spaces and their generalisations, the lattice-ordered Abelian groups that concern us here. (For background on Riesz spaces and ℓ-groups see [6] and [3, 5].) One main line of research concentrated on extending one given structure G to a minimal completion that enjoys the principal projection property, where Freudenthal’s theorem therefore applies. Such an extension is called *the projectable hull* of G. Here we present a new construction of the projectable hull of an Archimedean ℓ-group equipped with a strong order unit u that does not use direct limits, nor essential closures. Our construction exposes instead the intimate connection between projectable hulls and zero-dimensional compactifications of spectral spaces of minimal prime ideals.

In this first talk, we give the basic definitions concerning the notions involved in the construction. We topologize both the space of the maximal prime ideals $\mathrm{Max}\, G$ and the space of the minimal prime ideals $\mathrm{Min}\, G$ using the *spectral topology.* The closed sets for this topology are given by subsets of the form

$$\mathbb{V}_M(A) := \{\mathfrak{m} \in \mathrm{Max}\, G \colon A \subseteq \mathfrak{m}\}$$

and

$$\mathbb{V}_m(A) := \{\mathfrak{p} \in \mathrm{Min}\, G \colon A \subseteq \mathfrak{p}\},$$

as A ranges over arbitrary subsets of G. The space $\mathrm{Max}\, G$ is a compact Hausdorff space; the space $\mathrm{Min}\, G$ is a Hausdorff zero-dimensional space that need not be compact, [1].

It is well known that $\mathrm{Min}\, G$ is canonically thrown onto $\mathrm{Max}\, G$, as follows. Given $\mathfrak{p} \in \mathrm{Min}\, G$, a standard argument shows that, by virtue of the presence of the (strong) unit u, there exists at least one $\mathfrak{m}_{\mathfrak{p}} \in \mathrm{Max}\, G$ such that $\mathfrak{p} \subseteq \mathfrak{m}_{\mathfrak{p}}$. Since the prime ideals of G form a root system under set-theoretic inclusion, such an $\mathfrak{m}_{\mathfrak{p}}$ must be unique. Hence there is a continuous surjection $\lambda \colon \mathrm{Min}\, G \twoheadrightarrow \mathrm{Max}\, G$ defined by

$$\lambda \colon \mathfrak{p} \mapsto \mathfrak{m}_{\mathfrak{p}}.$$

The composition of λ with the Yosida representation of G ($\widehat{\cdot} \colon G \hookrightarrow \mathrm{C}(\mathrm{Max\, G})$, [8]) embeds G as a unital ℓ-subgroup into the ℓ-group $\mathrm{C}(\mathrm{Min\, G})$ of continuous functions $\mathrm{Min}\, G \to \mathbb{R}$ under pointwise operations: $g \in G$ is sent to the function $\widehat{g} \circ \lambda \colon \mathfrak{p} \in \mathrm{Min}\, G \mapsto (\widehat{g} \circ \lambda)(\mathfrak{p}) = \widehat{g}(\mathfrak{m}_{\mathfrak{p}}) \in \mathbb{R}$.

We focus now our attention on the case where $\mathrm{Min}\, G$ is compact. (Equivalent conditions to compactness are given in [2,7].) Here, the base $\{\mathbb{V}_m(g)\}_{g\in G}$ for the closed sets of $\mathrm{Min}\, G$ is a Boolean algebra under set-theoretic union, intersection and complementation. This base also coincides with the set of all clopen subsets of $\mathrm{Min}\, G$. Let $\mathrm{K}(\mathrm{Min\,G})$ be the collection of all characteristic functions on $\mathrm{Min}\, G$ – i.e. continuous maps $\mathrm{Min}\,(G) \to \mathbb{R}$ whose range is contained in $\{0, 1\}$. Then the elements in $\mathrm{K}(\mathrm{Min\,G})$ are precisely those functions whose value is 1 at each point of some $\mathbb{V}_m(g)$ (with $g \in G$), and 0 otherwise. Let $\widetilde{G}$ be the image of G under the map $\widehat{\cdot} \circ \lambda$, and let $\mathcal{P}(G)$ be the ℓ-subgroup of $\mathrm{C}(\mathrm{Min\,G})$ generated by $\widetilde{G} \cup \mathrm{K}(\mathrm{Min\,G})$. Our main result is the following:

Theorem 1 *The embedding*

$$\begin{aligned} \pi \colon G &\hookrightarrow \mathcal{P}(G) \\ g &\mapsto \widehat{g} \circ \lambda \end{aligned}$$

is the projectable hull of G.

For the generalization of our result to the case where $\mathrm{Min}\, G$ is not compact, please see Daniel McNeill's abstract.

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SEVENTH FRAMEWORK PROGRAMME MARIE CURIE ACTIONS

Stone duality above dimension zero: Infinitary algebras of real-valued functions on compact Hausdorff spaces

Luca Reggio

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Joint work with Vincenzo Marra

We shall give a finite axiomatisation of a class of infinitary algebras that we call δ-algebras. These are MV-algebras with one additional infinitary operation. We prove that the category of δ-algebras is a full subcategory of the category of MV-algebras, and it is dually equivalent to the category of compact Hausdorff spaces and continuous maps. In other words, an MV-algebra can be represented as the algebra of all $[0, 1]$-valued continuous functions on a compact Hausdorff space if, and only if, it is a δ-algebra. Equivalently, δ-algebras provide an (infinitary) equational axiomatisation of the class of unit balls of real commutative unital C^*-algebras.

This solves a long-standing problem that has been tackled by Pelletier, Isbell, Rosický and others. If time allows, we will provide context.

Lattice-ordered abelian groups and perfect MV-algebras: a topos-theoretic perspective

Anna Carla Russo
University of Salerno

This talk is based on [2]. We establish, generalizing Di Nola and Lettieri's categorical equivalence [3], a Morita-equivalence between the theory of lattice-ordered abelian groups and that of perfect MV-algebras. Further, after observing that the two theories are not bi-interpretable in the classical sense, we identify, by considering appropriate topos-theoretic invariants on their common classifying topos according to the 'bridge technique' of [1], three levels of bi-interpretability holding for particular classes of formulas: irreducible formulas, geometric sentences and imaginaries. Lastly, by investigating the classifying topos of the theory of perfect MV-algebras, we obtain various results on its syntax and semantics also in relation to the cartesian theory of the variety generated by Chang's MV-algebra, including a concrete representation for the finitely presentable models of the latter theory as finite products of finitely presentable perfect MV-algebras. Among the results established on the way, we mention a Morita-equivalence between the theory of lattice-ordered abelian groups and that of cancellative lattice-ordered abelian monoids with bottom element.

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A(nother) duality for the whole variety of MV-algebras

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Joint thinking with V. Marra and A. Pedrini.

Given a category C one can form its *ind-completion* by taking all formal directed colimits of objects in C. The "correct" arrows to consider are then families of some special equivalence classes of arrows in C ([1, V.1.2, pag. 225]). The *pro-completion* is formed dually by taking all formal directed limits. For general reasons, the ind-completion of a category C is dually equivalent to the *pro*-completion of the dual category C^{op}.

$$\text{ind-}C \simeq (\text{pro-}(C^{\mathrm{op}}))^{\mathrm{op}}. \tag{1}$$

Ind- and pro- completions are very useful objects (as they are closed under directed (co)limits) but cumbersome to use, because of the involved definitions of arrows between objects. We prove that if C is an algebraic category, then the situation considerably simplifies.

If V is any variety of algebras, one can think of any algebra A in V as colimit of finitely presented algebras as follows. Consider a presentation of A i.e., a cardinal μ and a congruence θ on the free μ-generated algebra $\mathcal{F}(\mu)$ such that $A \cong \mathcal{F}(\mu)/\theta$. Now, consider the set $F(\theta)$ of all finitely generated congruences contained in θ, this gives a directed diagram in which the objects are the finitely presented algebras of the form $\mathcal{F}(n)/\theta_i$ where $\theta_i \in F(\theta)$ and $X_1, ..., X_n$ are the free generators occurring in θ_i. It is straightforward to see that this diagram is directed, for if $\mathcal{F}(m)/\theta_1$ and $\mathcal{F}(n)/\theta_2$ are in the diagram, then both map into $\mathcal{F}(m+n)/\langle\theta_1 \uplus \theta_2\rangle$, where $\langle\theta_1 \uplus \theta_2\rangle$ is the congruence generated by the disjoint union of θ_1 and θ_2. Now, the colimit of such a diagram is exactly A. Denoting by V_{fp} the full subcategory of V of finitely presented objects, the above reasoning entails

$$V \simeq \text{ind-}V_{\mathrm{fp}}. \tag{2}$$

We apply the above machinery to the special case where V is the class of MV-algebras. One can then combine the duality between finitely presented MV-algebras and the category $P_{\mathbb{Z}}$ of rational polyhedra with $\mathbb{Z}$-maps [2], with (1) and (2) to obtain,

$$MV \simeq \text{ind-}MV_{\mathrm{fp}} \simeq \text{pro-}(P_{\mathbb{Z}})^{\mathrm{op}}. \tag{3}$$

This gives a categorical duality for the whole class of MV-algebras whose geometric content may be more transparent than other dualities in literature. In increasing order of complexity one has that an MV-algebra A:

1. is dual to a polyhedron (if A is finitely presented);
2. is dual to an intersection of polyhedra (if A is semisimple);
3. is dual to a countable nested sequence of polyhedra (if A is finitely generated);
4. is dual to the directed limit of a family of polyhedra. (if A is in none of the above cases).

References

[1] P. T. Johnstone. *Stone spaces.* Cambridge Univ. Pr., 1986.

[2] V. Marra and L. Spada. The dual adjunction between MV-algebras and Tychonoff spaces. *Studia Logica (Special issue dedicated to the memory of Leo Esakia)*, 100(1-2):253–278, 2012.

SEVENTH FRAMEWORK PROGRAMME
MARIE CURIE ACTIONS

Duality for sheaf representations of distributive lattices

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Joint work with M. Gehrke.

In this talk, I will report on ongoing joint work with Mai Gehrke, in which we study the following general duality-theoretic question.

Question. Let A be a distributive lattice with dual Priestley space X. Let F be a sheaf representation of A over a stably compact space. What is the additional structure on X that corresponds to the structure of the sheaf F?

We answer this question for sheaf representations of A that are c-soft, i.e., any section over a compact set can be extended to a global section. We also discuss how this theory applies in particular to MV-algebras, for which also see our recent joint paper with Vincenzo Marra [1].

References

[1] M. Gehrke and S. J. van Gool and V. Marra, Sheaf representations of MV-algebras and lattice-ordered abelian groups via duality. Journal of Algebra **417**, (2014) 290–332.