Categorical equivalences of classes of MTL-algebras through cancellative hoops

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Hoops have been widely used in relation with algebraic structures of many-valued logics, mainly in the framework of MTL-algebras. In this work we focus on cancellative hoops and we use them to build up some classes of MTL-algebras, giving a common framework to old and new results.

Indeed, starting from a cancellative hoop we can make three kind of operations that lead to the construction of different algebraic structures (see [2], [3], [4], [5]):

- If we add a bottom element to a cancellative hoop we obtain a product algebra and every directly indecomposable product algebra is of this kind.
- If we make a disconnected rotation of a cancellative hoop we have an MV-algebra belonging to the variety $V(C)$ generated by the Chang MV-algebra, and again all directly indecomposable MV-algebras in this variety are the disconnected rotation of a cancellative hoop.
- Finally, if we make the connected rotation of a cancellative hoop we obtain an IMTL-algebra in the variety $\mathbb{J}\Pi$ generated by the standard IMTL-algebra given by the Jenei rotation of the product t-norm, and every directly indecomposable IMTL-algebra in this variety is either a connected or a disconnected rotation of a cancellative hoop.

Using such results on directly irreducible algebraic structures, we can state a categorical equivalence between the category of product algebras and the category of MV-algebras belonging to the variety $V(C)$. Further we can consider the category Δ whose objects are pairs (A, F) made by an MV-algebra A in $V(C)$ and a filter of the Boolean skeleton $B(A)$ of A, and which arrows $f : (A, F) \to (A', F')$ are made by an MV-homomorphism $f : A \to B$ such that $f(F) \subseteq F'$. Using the characterization of free objects in the corresponding varieties, we can hence show that the category of finitely generated $\mathbb{J}\Pi$-algebras is equivalent to the full subcategory of Δ in which the objects are made by a finitely generated MV-algebra and a principal filter of the Boolean skeleton. Starting from this, taking filtered colimits in the corresponding categories, we show that the category of $\mathbb{J}\Pi$-algebras is equivalent to the category Δ.

The variety $\mathbb{J}\Pi$ has deep relations with MV-algebras: indeed MV-algebras in $\mathbb{J}\Pi$ form a variety that coincides with the variety generated by the three element MV-chain and by Chang's MV-algebra. As it is shown in [1], this allows to prove that the variety of MV-algebras cannot be axiomatised by one variable axioms starting from IMTL and hence, also the axiomatisation of MV from MTL and from BL cannot use only one variable From another perspective, $\mathbb{J}\Pi$ can be seen as the smallest standard generated variety of MTL-algebras containing the variety $V(C)$.

References

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