

Full Lambek Calculus with Contraction is Undecidable

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Besides the cut rule, Gentzen's sequent calculus **LJ** for propositional intuitionistic logic contains other structural rules, namely the rule of contraction (c), exchange (e), left weakening (i) and right weakening (o). By removing all these rules from **LJ**, one arrives at the full Lambek calculus **FL**. More generally, every extension of **FL** by a subset of the rules (c), (e), (i) and (o) defines a logic between **FL** and **LJ**. In [1] these logics are called *basic substructural logics*. It is known that each of these logics has an analytic sequent calculus. In particular, the cut rule is eliminable in all these calculi if the contraction rule is introduced in its global variant (for survey see e.g. [1, Chapter 4]).

Cut elimination is closely related to decidability. It is known that all basic substructural logics are decidable except of **FL_c** and **FL_{co}** where the former is the extension of **FL** by the contraction rule and the latter is the extension of **FL_c** by the right weakening rule. The decidability of basic substructural logics without the contraction rule follows immediately from the cut elimination theorem and is proved in [5]. On the other hand, such an easy argument is not applicable for logics with the contraction rule since this rule makes the proof-search tree infinite. Nevertheless, intuitionistic logic is decidable [2, 3] and the same holds for the extension of **FL** by the exchange and contraction rule [4] (the original combinatorial idea from the proof goes back to Kripke [6]). In contrast, we show that **FL_c** and **FL_{co}** are the only undecidable logics among all basic substructural logics. In fact, we prove that their common positive fragment **FL_c⁺** is undecidable.

References

- [1] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, April 2007.
- [2] Gerhard Gentzen. Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, 39(1):176–210, 1935.
- [3] Gerhard Gentzen. Untersuchungen über das logische Schließen II. *Mathematische Zeitschrift*, 39(1):405–431, 1935.
- [4] Eiji Kiriya and Hiroakira Ono. The contraction rule and decision problems for logics without structural rules. *Studia Logica*, 50(2):299–319, 1991.
- [5] Yuichi Komori. Predicate logics without the structural rules. *Studia Logica*, 45(4):393–404, 1986.
- [6] Saul Aaron Kripke. The problem of entailment. *Journal of Symbolic Logic*, 24:325, 1959. abstract.