Finite GBL-algebras and Heyting algebras with equivalence relations

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Our work attempts to describe models of fuzzy logics making use of crisp logics and indiscernibility relations.

We want to generalize the work that, in the algebraic context, was done for instance in [2], where some classes of MV-algebras are represented by Boolean algebras equipped with a distinguished automorphism, in [3], where all MV-algebras are represented by Boolean algebras with an equivalence relation given by a subgroup of the automorphism group, and in [5], where a similar representation is given for MV-algebras and BL-algebras.

We follow the approach of [3], but we take Heyting algebras in place of Boolean algebras, in the spirit of [5]. Namely, we consider a Heyting algebra, counterpart of an intuitionistic propositional theory, and we define an *indiscernibility relation* on it. We make two hypothesis on this relation:

- it is an equivalence relation given by a subgroup G of the automorphism group of the Heyting algebra: two elements are indiscernible if there is an automorphism in G mapping one to the other;
- if two chains are element-wise indiscernible, then there is an unique automorphism in G bringing one chain to the other.

In analogy with the BG-pairs mentioned in [3], we call *HG-pair* the pair made of a Heyting algebra and an indiscernibility relation. We describe the category of finite HG-pairs and we show that it is dually equivalent to a category WP^* of weighted partially ordered sets and open maps respecting weights in a certain sense.

Finite HG-pairs are also in correspondence with finite *GBL-algebras*. Commutative, integral GBL-algebras are defined as commutative, integral, divisible residuated lattices, or equivalently as hoops with a lattice reduct. The variety of commutative, integral GBL-algebras extends the variety of MV-algebras and the one of Heyting algebras and it can be viewed as a fuzzy generalization of the latter. Informally, GBL-algebras generalize Heyting algebras in a similar way as MV-algebras generalize Boolean algebras.

Finite GBL-algebras are always commutative and, as shown in [4], they can be represented as poset products of Wajsberg chains. This result brings a natural correspondence between the category of finite GBL-algebras with homomorphisms and a category of weighted posets whose class of morphisms extends the one of WP^*. A similar representation for the subclass of finite BL-algebras is given in [1].

We retrace the duality of finite GBL-algebras and weighted posets adding that little bit of structure needed to recover a category of algebras equivalent to the category of finite HG-pairs. If we add to finite GBL-algebras a further binary operation $\oplus$, which behaves like the sum in the subclass of MV-algebras and like the disjunction in the subclass of Heyting algebras, we have a category of algebras dual to WP^*. Thus we have an equivalence between the category of what we call *finite* $GBL_\oplus$*-algebras* and the category of finite HG-pairs.

Finite $\text{GBL}_\oplus$-algebras are a *wide* subcategory of finite GBL-algebras, having the same class of objects and a subclass of morphisms. Indeed, if we define the lower and upper approximation of an element by idempotents, morphisms of GBL-algebras preserve only the lower approximation, while morphisms of $\text{GBL}_\oplus$-algebras preserve both.

References

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