Dipartimento di Statistica
"Giuseppe Parenti"

WORKING PAPER 2004/03

Small Area Estimation by Spatial Models: the Spatial Empirical Best Linear Unbiased Prediction (Spatial EBLUP)

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Abstract

The methods used for Small Area Estimation can be classified by the type of inference in design based (direct domain estimators), model assisted (synthetic and composite estimators) and model based (small area models). This paper deals the small area models and introduces spatial dependence among the random area effects. In fact, especially in most of applications on environmental data, it should be more reasonable to assume that the random effects between the neighboring areas are correlated and the correlation decays to zero as distance increases.

Considering the spatial dimension of the data, a model with spatially autocorrelated errors has to be implemented, a modified estimator (Spatial EBLUP) is provided and the Mean Square Error (MSE) estimator is obtained. The empirical analysis is carried out on some simulated experiments and the results show an appreciable improvement of the statistical properties of the small area estimators.

KEY WORDS: Small area models; spatial models; EBLUP; spatial EBLUP; mean square error estimator.

1 Design Based, Model Assisted and Model Based Methods

Sample survey data are extensively used to provide reliable direct estimates of totals and means for the whole population and large areas or domains.

Domains may be defined by geographic areas (state, county, municipality, school district, etc.) or socio-demographic groups or other sub-populations (age-sex-race group within a large geographic area). A domain is regarded as "*small*" if the domain-specific sample is not large enough to support direct estimates of adequate precision; they are likely to yield large standard errors due to the unduly small size of the sample in the area (Ghosh and Rao, 1994). The demand of reliable statistics for small areas, when only reduced sizes of the samples are available, has promoted the development of statistical methods from both the theoretical and empirical point of view.

In Italy, as in many other counties, there is a growing need for current and reliable data on small areas. This information need concerns most sample surveys realized by the National Statistical Institute (ISTAT), especially the Labour Force Survey (LFS), which has been studied to warrant accuracy in regional estimates (Falorsi *et al.*, 1994) .

In Great Britain devolution for Scotland, Wales and Northern Ireland, and the creation of nine Regional Development Agencies in England, has stimulated demand for information for each of these areas, as well as more detail within these.

In the United States the Census Bureau and a consortium of other Federal Agencies have started a project to provide post-censal estimates of income and poverty for small areas during the 1990's.

The methods used for SAE can be classified by the type of inference:

1. **Design based**: they make use of survey weights and the associated inferences are based on the probability distribution induced by the sampling design with the population values held fixed. The design based direct domain estimators are design based.

2. **Model assisted**: they make use of *working* models and are also design based, aiming at making the inferences "robust" to possible model misspecification. The role of the model is to describe the finite population point scatter, even if the assumption is never made that the population was really generated by the model. The basic property and the conclusion about finite population parameters are therefore independent of model assumptions. These procedures are thus model assisted, but they are not model dependent (Särdnal *et al.*, 1992). The generalized regression estimator (GREG), synthetic and composite estimators are model assisted.

3. **Model based**: the methods start from a specification of a model for the N-dimensional distribution of $Y = (Y_1, Y_2, ..., Y_N)$, where Y_k is a random variable tied to the k-th element. This model is denoted ξ and is called a superpopulation model. The actual finite population vector, $y = (y_1, y_2, ..., y_N)$, is considered to be a realization of Y. A sample s has been drawn. A model based inference is interpreted by visualizing a long series of realizations of the finite population vector Y for the fixed s. The inference is tied to the particular s that was realized, and not to other samples (Särdnal *et al.*, 1992). Small area models are model based.

In estimation for small areas the direct survey estimates often have large sampling variability, then it is common to borrow strength from other small areas. From this point of view the potentialities of Geographical Information System (GIS) as a tool for the compilation of statistics, particularly in the field of small area statistics, are large. For example, especially in most of applications on environmental data, the land use, the quote, the slope can be used as auxiliary information to estimate a mean in the small areas. The use of GIS is due to implicit conviction that the data of neighboring areas are correlated and the correlation decays to zero as distance increases. But, in the traditional small area models the random area effects are considered as independent (Section 2). Considering the spatial dimension of the data (Section 3), the paper introduces spatial dependence among the random area effects, then a model with spatially autocorrelated errors has to be implemented and a modified estimator (Spatial EBLUP) is provided; moreover the Mean Square Error (MSE) estimator is obtained (Section 4). The empirical analysis is carried out on some simulated experiments and the results show an appreciable improvement of the statistical properties of the small area estimators (Section 5). In Section 6 some concluding remarks are made.

2 Small Area Models

Small area models make use of explicit linking models based on random area-specific effects that account for between areas variation beyond that is explained by auxiliary variables included in the model (Pfeffermann, 2002).

Inferences from model based estimators refer to the distribution implied by the assumed model. Model selection and validation play an important role in model based estimation, in fact if the assumed models do not perform a good fit to the data, the estimators will be model biased and can lead to wrong inferences.

Some statisticians see the small area models as a methodological advantage. Optimal estimators can be derived under the assumed model. Model diagnostic can be used to find suitable model that fit the data well. Area-specific measures of precision can be associated with each small area estimate. Some models can be considered according to the nature of the response variables and the complexity of data structures (Rao, 2003).

Other statisticians consider wrong to carry out model based estimation because they believe artificial to visualize repeated finite populations because actually there is only one finite population.

Small area models can be classified into two broad types:

- **area level random effect models**, which are used when auxiliary information is available only at area level. They relate small area direct estimators to area-specific covariates (Fay and Herriot, 1979);
- **nested error unit level regression models**, employed originally by Battese *et al.* (1988) for predicting areas under corn and soybeans in 12 counties of the state of Iowa in the U.S (Pfeffermann, 2002). These models relate the unit values of a study variable to unit-specific covariates.

The area basic model considers the random area effects as independent. In practice, it should be more reasonable to assume that the random effects between the neighboring areas (for instance the neighborhood could be define by a distance criterium) are correlated and the correlation decays to zero as distance increases. Considering the spatial dimension of the data, a model with spatially autocorrelated errors has to be implemented, as it is shown in the next sections.

3 Spatial Models

In the simplest sense, spatial statistics is the analysis of spatial pattern in data. In spatial statistics is necessary to do a distinction between methods that are concerned with visualizing spatial data, those which are exploratory, summarizing and investigating map patterns and relationships, and those which rely on the specification of a statistical spatial model and the estimation of its parameters. In this section, this last case will be treated, modelling spatial interaction, that, commonly means incorporating the spatial dependence into the covariance structure via an autoregressive model.

The spatial data can be showed as follow: the study area is given by D; it is usually a subset of two-dimensional space. A vector $\mathbf{s}$ contains information on the data location such as latitude and longitude, and at location $\mathbf{s}$ the observed value y is obtained: the $\mathbf{Y}(\mathbf{s})$ is a random variable at each location. Then the general spatial model can be expressed as: $\{\mathbf{Y}(\mathbf{s}) \ : \ \mathbf{s} \in D\}$ where $D \subset \Re^d$ and d is the dimension (usually $d = 2$).

The basic spatial models can be distinguished as follows:

1. *geostatistical data*: D is a continuous subset of $\Re^d$; $\mathbf{Y}(\mathbf{s})$ is a random vector at location $\mathbf{s} \in D$ (example of geostatistical data: temperature values taken at weather stations);
2. *lattice data*: D is a fixed but countable subset of $\Re^d$ such as a grid some representation with nodes; $\mathbf{Y}(\mathbf{s})$ is a random vector at location $\mathbf{s} \in D$ (example of lattice data: counts of disease in a county);
3. *point patterns*: D is a random subset of $\Re^d$ and it is called a point process; if $\mathbf{Y}(\mathbf{s})$ is a random vector at location $\mathbf{s} \in D$ then it is a *marked spatial point process*; if $\mathbf{Y}(\mathbf{s}) \equiv 1$ so that it is a degenerate random variable, then only D is random and it is called a *spatial point process* (example of point pattern data: locations of trees in a forest).This model was first formulated by Cressie (1993).

In small area estimation irregular lattice data are dealt, in fact the data collected at a given location represent a summary for the subregion in which the location is sited. In this case two autoregressive models are commonly employed: the simultaneously autoregressive model (SAR) and the conditional autoregressive model (CAR). These models produce spatial dependence in the covariance structure as a function of a neighborhood matrix $\mathbf{W}$ and a fixed unknown spatial correlation parameter (Wall, 2004).

In general the behavior of spatial phenomena is often the result of a mixture of both first order and second order effects. First order effects relate to the variation in the mean value of the process in the space (a global or large scale trend). Second order effects result from the spatial correlation structure, or the spatial dependence in the process; in other words, the tendency for deviations in value of the process from its mean to follow each other in neighboring sites (local or small scale effects).

These two effects can be modelled using the general regression model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{v} \tag{1}$$

with $\mathbf{v} \sim (\mathbf{0}, \mathbf{C})$ and $\mathbf{C}$ is specified by an interaction scheme including, in the model, relationships between variables and their neighboring values usually involving a few extra parameters which need to be estimated and which indirectly specify particular form of $\mathbf{C}$. Such models need not to assume stationarity for the second order component (Bailey and Gatrell, 1995).

The SAR model is based on the specification of how data at the various sites interact simultaneously and it can be expressed as:

$$\begin{aligned} \mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{v} \\ \mathbf{v} &= \rho\mathbf{W}\mathbf{v} + \mathbf{u} \end{aligned} \tag{2}$$

where $\mathbf{X}\boldsymbol{\beta}$ is the large scale, non-spatial trend surface which depends on covariates, $\mathbf{v}$ is the second order variation with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{C} = \sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}$, ρ is the spatial dependence parameter, it is called spatial autoregressive coefficient, $\mathbf{u}$ is a vector of error terms which are independent with zero mean and constant variance σ_u^2, $\mathbf{W}$ is the proximity matrix and $\mathbf{I}$ is an identity matrix. The variate interaction models, as mentioned above, indirectly imply a covariance structure $\mathbf{C}$, that in this case it can be derived directly with some matrix manipulation:

$$\begin{aligned} \mathbf{C} = E[\mathbf{v}\mathbf{v}^T] &= E\left[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{u}\mathbf{u}^T\left((\mathbf{I} - \rho\mathbf{W})^{-1}\right)^T\right] = \\ &= (\mathbf{I} - \rho\mathbf{W})^{-1}E[\mathbf{u}\mathbf{u}^T]\left((\mathbf{I} - \rho\mathbf{W})^{-1}\right)^T = \\ &= (\mathbf{I} - \rho\mathbf{W})^{-1}\sigma_u^2\mathbf{I}\left((\mathbf{I} - \rho\mathbf{W})^{-1}\right)^T = \\ &= \sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\left((\mathbf{I} - \rho\mathbf{W})^T\right)^{-1} = \\ &= \sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}. \end{aligned} \tag{3}$$

The CAR model is very different from SAR model: the conditional model assumes that the probability of observing a particular value at a given site is a conditional probability, i.e. it depends on the value of Y in the neighborhood of the site. The SAR model states that the probability is a product of functions which can not be interpreted as conditional probabilities. The CAR model can be written as:

$$\begin{aligned} \mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{v} \\ \mathbf{v} &\sim (\mathbf{0}, \sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}). \end{aligned} \tag{4}$$

Then the most difference between SAR and CAR model is in the covariance structure. In CAR model $\mathbf{W}$ needs to be symmetrical and $(\mathbf{I} - \rho\mathbf{W})$ needs to be strictly positive definite to ensure the existence and symmetry of $\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}$ in the conditional scheme (Upton and Fingleton, 1985).

The proximity matrix $\mathbf{W}$ indicates whether the areas are neighbor or not. One common way to do this is to define $w_{ij} = 1$ if region i shares a common edge or border with region j or 0 otherwise. There are other ways to define $\mathbf{W}$ as restricting rows of the neighborhood matrix to sum to 1 or creating more elaborate weights as functions of the length of borders (Wall, 2004).

4 Spatial Area Level Random Effect Models

In order to take into account the correlation between neighboring areas we regarded to the spatial models and how these models could be utilized in small area estimation (Cressie, 1991). In the standard linear regression model, spatial dependence can be incorporated in two distinct ways: it can be specified a regression which has autoregressive terms, or it can be formulated a regression model with spatially autocorrelated residuals (Anselin, 1992). In this study a standard linear regression is considered and the spatial dependence has been incorporated in the error structure ($E[v_i, v_j] \neq 0$). It can be specified in a number of different ways, and results in a error variance covariance matrix of the form:

$$E[v_i, v_j] = \mathbf{\Omega}(\boldsymbol{\tau}), \tag{5}$$

where $\boldsymbol{\tau}$ is a vector of parameters, such as the coefficient in a Simultaneously Autoregressive (SAR) or Conditional Autoregressive (CAR) error process, and v_i, v_j are the area random effects.

4.1 SAR Model

A SAR error model (2) includes random area effects and the area specific auxiliary covariates $\mathbf{x}_i = (x_{i,1}, x_{i,2}, ..., x_{i,p})$ are related to the parameters of inferential interest θ_i (totals y_i, means $\bar{y}_i$); in matrix form it is:

$$\boldsymbol{\theta} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} \tag{6}$$

where $\boldsymbol{\beta}$ is the regression parameters vector $p\times 1$, $\mathbf{Z}$ is a matrix of known positive constants, $\mathbf{v}$ is the second order variation (2) with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{C} = \sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}$; ρ, $\mathbf{u}$ and $\mathbf{W}$ are defined as in above section. Moreover it assumes that direct estimators $\hat{\theta}_i$ are available and design-unbiased:

$$\hat{\theta}_i = \theta_i + e_i \tag{7}$$

where e_i are independent sampling errors with mean 0 and known variance ψ_i. Combining (6) and (7) a new model with spatially correlated errors could be implemented and in matrix form it is:

$$\hat{\boldsymbol{\theta}} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{u} + \mathbf{e} \tag{8}$$

Spatial models are a special case of the general linear mixed model. The covariance matrices $m \times m$ of $\mathbf{v}$ and $\mathbf{e}$ are:

$$\mathbf{G} = \sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1} \tag{9}$$

and

$$\mathbf{R} = diag(\psi_i). \tag{10}$$

Then the covariance matrix of the studied variable is:

$$\mathbf{V} = \mathbf{R} + \mathbf{Z}\mathbf{G}\mathbf{Z}^T = diag(\psi_i) + \mathbf{Z}\sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T \tag{11}$$

with $\mathbf{v}$ and $\mathbf{e}$ independently distributed. The Spatial Best Linear Unbiased Predictor (Spatial BLUP) estimator of θ_i is:

$$\tilde{\theta}_i^S(\sigma_u^2, \rho) = \mathbf{x}_i\hat{\boldsymbol{\beta}} + \mathbf{b}_i^T\{\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\}\mathbf{Z}^T\{diag(\psi_i)+\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\}^{-1}(\hat{\boldsymbol{\theta}}-\mathbf{X}\hat{\boldsymbol{\beta}}) \tag{12}$$

where $\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}\hat{\boldsymbol{\theta}}$ and $\mathbf{b}_i^T$ is $1 \times m$ vector $(0, 0, ...0, 1, 0, ...0)$ with 1 in the i-th position.

The $MSE[\tilde{\theta}_i^S(\sigma_u^2, \rho)]$, depending on two parameters (σ_u^2, ρ), can be expressed as:

$$MSE[\tilde{\theta}_i^S(\sigma_u^2, \rho)] = g_{1i}(\sigma_u^2, \rho) + g_{2i}(\sigma_u^2, \rho) \tag{13}$$

with

$$g_{1i}(\sigma_u^2,\rho)=\mathbf{b}_i^T\{\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}-\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\times$$
$$\times\{diag(\psi_i)+\mathbf{Z}\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\}^{-1}\mathbf{Z}\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\}\mathbf{b}_i \quad (14)$$

and

$$\begin{gathered}g_{2i}(\sigma_u^2,\rho)=(\mathbf{x}_i-\mathbf{b}_i^T\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\\ \{diag(\psi_i)+\mathbf{Z}\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\}^{-1}\mathbf{X})\times\\ \times(\mathbf{X}^T\{diag(\psi_i)+\mathbf{Z}\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\}^{-1}\mathbf{X})^{-1}\times\\ (\mathbf{x}_i-\mathbf{b}_i^T\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\\ \{diag(\psi_i)+\mathbf{Z}\sigma_u^2[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]^{-1}\mathbf{Z}^T\}^{-1}\mathbf{X})^T.\end{gathered} \quad (15)$$

where the first term $g_{1i}(\sigma_u^2,\rho)$ is due to the estimation of random effects and the second term $g_{2i}(\sigma_u^2,\rho)$ is due to the estimation of $\boldsymbol{\beta}$ (Rao, 2003).

4.1.1 Spatial EBLUP.

The estimator $\tilde{\theta}_i^S(\sigma_u^2,\rho)$ depends on the variance components σ_u^2 and ρ, but in practice they will be unknown. Replacing the parameters with asymptotically consistent estimators $\hat{\sigma}_u^2$, $\hat{\rho}$, a two stage estimator $\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})$ is obtained and it is called Spatial EBLUP:

$$\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})=\mathbf{x}_i\hat{\boldsymbol{\beta}}+\mathbf{b}_i^T\{\hat{\sigma}_u^2[(\mathbf{I}-\hat{\rho}\mathbf{W})(\mathbf{I}-\hat{\rho}\mathbf{W}^T)]^{-1}\}\mathbf{Z}^T\{diag(\psi_i)+\hat{\sigma}_u^2[(\mathbf{I}-\hat{\rho}\mathbf{W})(\mathbf{I}-\hat{\rho}\mathbf{W}^T)]^{-1}\}^{-1}(\hat{\boldsymbol{\theta}}-\mathbf{X}\hat{\boldsymbol{\beta}}) \quad (16)$$

with $\mathbf{b}_i^T=(0,0,...0,1,0,...0)$ and 1 referred to i-th area. It has some properties:

1. it is unbiased for θ;
2. $E[\tilde{\theta}(\hat{\sigma}_u^2,\hat{\rho})]$ is finite;
3. $\hat{\sigma}_u^2,\hat{\rho}$ are any translation invariant estimators of σ_u^2 and ρ (Kackar and Harville, 1984).

Assuming normality, σ_u^2 and ρ can be estimated both by ML and REML procedures. The log-likelihood function is:

$$l(\boldsymbol{\beta},\sigma_u^2,\rho)=-\frac{1}{2}m\log 2\pi-\frac{1}{2}\log|\mathbf{V}|-\frac{1}{2}(\hat{\boldsymbol{\theta}}-\mathbf{X}\boldsymbol{\beta})^T\mathbf{V}^{-1}(\hat{\boldsymbol{\theta}}-\mathbf{X}\boldsymbol{\beta}) \quad (17)$$

with $\mathbf{V}$ as represented in (11) and the partial derivatives of $l(\boldsymbol{\beta},\sigma_u^2,\rho)$ with respect to σ_u^2 and ρ are given by

$$\begin{aligned}s_{\sigma_u^2}(\boldsymbol{\beta},\sigma_u^2,\rho)=\tfrac{\partial l}{\partial\sigma_u^2}= &-\tfrac{1}{2}tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\}-\tfrac{1}{2}(\hat{\boldsymbol{\theta}}-\mathbf{X}\boldsymbol{\beta})^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})(\hat{\boldsymbol{\theta}}-\mathbf{X}\boldsymbol{\beta})\\ s_{\rho}(\boldsymbol{\beta},\sigma_u^2,\rho)=\tfrac{\partial l}{\partial\rho}= &-\tfrac{1}{2}tr\{\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2[-\mathbf{C}^{-1}(2\rho\mathbf{W}\mathbf{W}^T-2\mathbf{W})\mathbf{C}^{-1}]\mathbf{Z}^T\}-\\ &-\tfrac{1}{2}(\hat{\boldsymbol{\theta}}-\mathbf{X}\boldsymbol{\beta})^T(-\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2[-\mathbf{C}^{-1}(2\rho\mathbf{W}\mathbf{W}^T-2\mathbf{W})\mathbf{C}^{-1}]\mathbf{Z}^T\mathbf{V}^{-1})(\hat{\boldsymbol{\theta}}-\mathbf{X}\boldsymbol{\beta})\end{aligned} \quad (18)$$

with $\mathbf{C}=[(\mathbf{I}-\rho\mathbf{W})(\mathbf{I}-\rho\mathbf{W}^T)]$. The matrix of expected second derivatives of $-l(\boldsymbol{\beta},\sigma_u^2,\rho)$ with respect to σ_u^2 and ρ is given by

$$\mathcal{I}(\sigma_u^2,\rho)=\begin{bmatrix}\frac{1}{2}tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\} & \frac{1}{2}tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}\mathbf{Z}^T\}\\ \frac{1}{2}tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}\mathbf{Z}^TV^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\} & \frac{1}{2}tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}\mathbf{Z}^T\}\end{bmatrix} \quad (19)$$

with $\mathbf{A}=\sigma_u^2[-\mathbf{C}^{-1}(2\rho\mathbf{W}\mathbf{W}^T-2\mathbf{W})\mathbf{C}^{-1}]$. The ML estimators $\hat{\sigma}_{u_{ML}}^2$ and $\hat{\rho}_{ML}$ can be obtained iteratively using the "scoring" algorithm:

$$\begin{bmatrix}\sigma_u^2\\ \rho\end{bmatrix}^{(n+1)}=\begin{bmatrix}\sigma_u^2\\ \rho\end{bmatrix}^{(n)}+[\mathcal{I}(\sigma_u^{2^{(n)}},\rho^{(n)})]^{-1}\cdot s\left[\hat{\boldsymbol{\beta}}(\sigma_u^{2^{(n)}},\rho^{(n)}),\sigma_u^{2^{(n)}},\rho^{(n)}\right] \quad (20)$$

where n indicates the number of iteration.

The ML procedure to estimate σ_u^2 and ρ does not consider the loss in degrees of freedom due to estimating $\boldsymbol{\beta}$. This drawback involves the use of REML method. The partial derivatives of the restricted log-likelihood function $l_R(\sigma_u^2, \rho)$ with respect to variance components are:

$$\begin{array}{ll} s_{R_{\sigma_u^2}}(\sigma_u^2,\rho) = \frac{\partial l_R}{\partial \sigma_u^2} = & -\frac{1}{2}tr\{\mathbf{PZC}^{-1}\mathbf{Z}^T\} + \frac{1}{2}\hat{\theta}^T\mathbf{PZC}^{-1}\mathbf{Z}^T\mathbf{P}\hat{\theta} \\ s_{R_\rho}(\sigma_u^2,\rho) = \frac{\partial l_R}{\partial \rho} = & -\frac{1}{2}tr\{\mathbf{PZ}\sigma_u^2[-\mathbf{C}^{-1}(2\rho\mathbf{WW}^T - 2\mathbf{W})\mathbf{C}^{-1}]\mathbf{Z}^T\} + \\ & +\frac{1}{2}\hat{\boldsymbol{\theta}}^T\mathbf{PZ}\sigma_u^2[-\mathbf{C}^{-1}(2\rho\mathbf{WW}^T - 2\mathbf{W})\mathbf{C}^{-1}]\mathbf{Z}^T\mathbf{P}\hat{\boldsymbol{\theta}} \end{array} \tag{21}$$

with $\mathbf{P} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}$ and $\mathbf{C} = [(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}^T)]$. The loss in degrees of freedom are taken into account in the REML method by using the transformed data $\boldsymbol{\theta}^* = \mathbf{F}^T\hat{\boldsymbol{\theta}}$, where $\mathbf{F}$ is any $n \times (m-p)$ matrix of full rank orthogonal to the $m \times p$ matrix $\mathbf{X}$. The $\mathcal{I}_R(\sigma_u^2, \rho)$ matrix assumes the form:

$$\mathcal{I}_R(\sigma_u^2,\rho) = \begin{bmatrix} \frac{1}{2}tr\{\mathbf{PZC}^{-1}\mathbf{Z}^T\mathbf{PZC}^{-1}\mathbf{Z}^T\} & \frac{1}{2}tr\{\mathbf{PZC}^{-1}\mathbf{Z}^T\mathbf{PZAZ}^T\} \\ \frac{1}{2}tr\{\mathbf{PZAZ}^T\mathbf{PZC}^{-1}\mathbf{Z}^T\} & \frac{1}{2}tr\{\mathbf{PZAZ}^T\mathbf{PZAZ}^T\} \end{bmatrix} \tag{22}$$

with $\mathbf{A} = \sigma_u^2[-\mathbf{C}^{-1}(2\rho\mathbf{WW}^T - 2\mathbf{W})\mathbf{C}^{-1}]$. Then the “scoring” algorithm (20) is used and at convergence the REML estimators are obtained and the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}_R$, $\hat{\sigma}^2_{u_R}$ and $\hat{\rho}_R$ has a diagonal structure $diag\left[\bar{\mathbf{V}}(\hat{\boldsymbol{\beta}}_R), \bar{\mathbf{V}}(\hat{\sigma}^2_{u_R}, \hat{\rho}_R)\right] \approx diag\left[\bar{\mathbf{V}}(\hat{\boldsymbol{\beta}}_{ML}), \bar{\mathbf{V}}(\hat{\sigma}^2_{u_{ML}}, \hat{\rho}_{ML})\right]$ with

$$\bar{\mathbf{V}}(\hat{\boldsymbol{\beta}}_R) \approx \bar{\mathbf{V}}(\hat{\boldsymbol{\beta}}_{\boldsymbol{ML}}) = (\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}$$

$$\bar{\mathbf{V}}(\hat{\sigma}^2_{u_R}, \hat{\rho}_R) \approx \bar{\mathbf{V}}(\hat{\sigma}^2_{u_{ML}}, \hat{\rho}_{ML}) = \mathcal{I}^{-1}(\sigma_u^2, \rho) \tag{23}$$

The ML and REML estimators are robust, in fact they may work well even under non normal distributions (Jiang, 1996).

The ML estimators can be calculated by the Newton Raphson algorithm:

$$\begin{bmatrix} \sigma_u^2 \\ \rho \end{bmatrix}^{(n+1)} = \begin{bmatrix} \sigma_u^2 \\ \rho \end{bmatrix}^{(n)} - \left\{\partial^2 l\left[\hat{\boldsymbol{\beta}}(\sigma_u^{2^{(n)}}, \rho^{(n)}), \sigma_u^{2^{(n)}}, \rho^{(n)}\right]\right\}^{-1} \bigtriangledown l\left[\hat{\boldsymbol{\beta}}(\sigma_u^{2^{(n)}}, \rho^{(n)}), \sigma_u^{2^{(n)}}, \rho^{(n)}\right] \tag{24}$$

where $\bigtriangledown l\left[\hat{\boldsymbol{\beta}}(\sigma_u^{2^{(n)}}, \rho^{(n)}), \sigma_u^{2^{(n)}}, \rho^{(n)}\right]$ is a 2×1 vector of partial derivatives and $\partial^2 l\left[\hat{\boldsymbol{\beta}}(\sigma_u^{2^{(n)}}, \rho^{(n)}), \sigma_u^{2^{(n)}}, \rho^{(n)}\right]$ is the Hessian matrix.

4.1.2 MSE of Spatial EBLUP.

The MSE of Spatial EBLUP estimator appears to be insensitive to the choice of the estimator $\hat{\sigma}_u^2$ and $\hat{\rho}$. Under normality of random effects

$$MSE[\tilde{\theta}_i(\hat{\sigma}_u^2)] = MSE[\tilde{\theta}_i(\sigma_u^2)] + E[\tilde{\theta}_i(\hat{\sigma}_u^2) - \tilde{\theta}_i(\sigma_u^2)]^2 \tag{25}$$

where the last term is obtained as an approximation because is generally intractable. Then an approximation to the $MSE[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})]$ is:

$$MSE[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})] \approx g_{1i}(\sigma_u^2, \rho) + g_{2i}(\sigma_u^2, \rho) + g_{3i}(\sigma_u^2, \rho) \tag{26}$$

where $g_{3i}(\sigma_u^2, \rho)$ is approximately

$$tr\left\{\begin{bmatrix} \mathbf{b}_i^T\left(\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{ZC}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\right) \\ \mathbf{b}_i^T\left(\mathbf{AZ}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{ZAZ}^T\mathbf{V}^{-1})\right) \end{bmatrix}\mathbf{V}\times\right.$$

$$\left.\times\begin{bmatrix} \mathbf{b}_i^T\left(\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{ZC}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\right) \\ \mathbf{b}_i^T\left(\mathbf{AZ}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{ZAZ}^T\mathbf{V}^{-1})\right) \end{bmatrix}^T \bar{\mathbf{V}}(\hat{\sigma}_u^2, \hat{\rho})\right\} \tag{27}$$

In practical application the estimator $\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})$ has to be associated with an estimator of $MSE[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})]$. An approximately unbiased estimator of this mean square error is computed using the following expression:

$$mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})] \approx g_{1i}(\hat{\sigma}_u^2, \hat{\rho}) + g_{2i}(\hat{\sigma}_u^2, \hat{\rho}) + 2g_{3i}(\hat{\sigma}_u^2, \hat{\rho}) \tag{28}$$

if $\hat{\sigma}_u^2$ and $\hat{\rho}$ are REML estimators. Otherwise, if ML procedure is used, the $mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})]$ is given by

$$mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho})] \approx g_{1i}(\hat{\sigma}_u^2, \hat{\rho}) - \mathbf{b}_{ML}^T(\hat{\sigma}_u^2, \hat{\rho}) \bigtriangledown g_{1i}(\hat{\sigma}_u^2, \hat{\rho}) + g_{2i}(\hat{\sigma}_u^2, \hat{\rho}) + 2g_{3i}(\hat{\sigma}_u^2, \hat{\rho}) \tag{29}$$

with

$$\bigtriangledown g_{1i}(\sigma_u^2, \rho) = \mathbf{b}_i^T \left\{ \begin{array}{l} (\mathbf{C}^{-1} - [\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{C}^{-1} + \sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{Z}\sigma_u^2\mathbf{C}^{-1} + \\ (\mathbf{A} - [\mathbf{A}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{C}^{-1} + \sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{Z}\sigma_u^2\mathbf{C}^{-1} + \end{array}\right.$$

$$\left. \begin{array}{l} +\sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}]) \\ +\sigma_u^2\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}]) \end{array} \right\} \mathbf{b}_i \tag{30}$$

and

$$\mathbf{b}_{ML}^T(\sigma_u^2, \rho) = \frac{1}{2m} \left\{ \mathcal{I}^{-1}(\sigma_u^2, \rho) \begin{bmatrix} tr[(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{X}] \\ tr[(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{A}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{X}] \end{bmatrix} \right\} \tag{31}$$

where $\mathcal{I}(\sigma_u^2, \rho)$ is given by (19). If the term $\mathbf{b}_{ML}^T(\hat{\sigma}_u^2, \hat{\rho}) \bigtriangledown g_{1i}(\hat{\sigma}_u^2)$ is ignored, the use of ML estimators could lead to underestimation of MSE approximation.

Sampling variances, ψ_i, may not be known in practice. Two possible situations are distinguishable:

1. the sampling design selects elementary units in all the small areas of interest (domains);

2. the sampling design does not control the selection in all of the domains, as a result of the random selection of the sample, some domains could be empty.

In case (1) the sampling variability of the direct sampling estimate can be estimated by an appropriate estimator induced by sampling design. The statistical properties of the estimator depends also on the sample size at the area level.

In case (2) Rao (1998) proposes to smooth the sampling error associated to population level estimator in order to estimate the sampling variability at the small area level.

The estimated variance $\hat{\psi}_i$ is then treated as a proxy to ψ_i. As result the $mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho}, \hat{\psi}_i)]$ is greater than $mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2, \hat{\rho}, \psi_i)]$.

4.2 CAR Model

An alternative to the SAR error model is the Conditional Autoregressive error model that introduces a new specification of spatially autocorrelated errors. The covariance matrix is set equal to $\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}$. Hence $\mathbf{W}$ needs to be symmetrical and $(\mathbf{I} - \rho\mathbf{W})$ needs to be strictly positive definite to ensure the existence and symmetry of $\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}$ in the conditional scheme (Upton and Fingleton, 1985).

The Spatial BLUP with CAR covariance matrix is:

$$\tilde{\theta}_i^C(\sigma_u^2, \rho) = \mathbf{x}_i\hat{\boldsymbol{\beta}} + \mathbf{b}_i^T\{\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\}\mathbf{Z}^T\{diag(\psi_i) + \sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\}^{-1}(\hat{\boldsymbol{\theta}} - \mathbf{X}\hat{\boldsymbol{\beta}}) \tag{32}$$

with $\mathbf{b}_i^T$ is as above. Following the same way conducted for SAR error model both ML and REML procedures are showed to estimate the variance components σ_u^2 and ρ, and the Spatial EBLUP is achieved:

$$\tilde{\theta}_i^C(\hat{\sigma}_u^2, \hat{\rho}) = \mathbf{x}_i\hat{\boldsymbol{\beta}} + \mathbf{b}_i^T\{\hat{\sigma}_u^2(\mathbf{I} - \hat{\rho}\mathbf{W})^{-1}\}\mathbf{Z}^T\{diag(\psi_i) + \hat{\sigma}_u^2(\mathbf{I} - \hat{\rho}\mathbf{W})^{-1}\}^{-1}(\hat{\boldsymbol{\theta}} - \mathbf{X}\hat{\boldsymbol{\beta}}). \tag{33}$$

4.2.1 Estimation of σ_u^2 and ρ.

The log-likelihood function is equal to the (17) and the partial derivatives of $l(\boldsymbol{\beta}, \sigma_u^2, \rho)$ with respect to σ_u^2 and ρ are given by

$$\begin{aligned} s_{\sigma_u^2}(\boldsymbol{\beta}, \sigma_u^2, \rho) = \frac{\partial l}{\partial \sigma_u^2} = & -\frac{1}{2} tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\} + \frac{1}{2}(\hat{\boldsymbol{\theta}} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})(\hat{\boldsymbol{\theta}} - \mathbf{X}\boldsymbol{\beta}) \\ s_{\rho}(\boldsymbol{\beta}, \sigma_u^2, \rho) = \frac{\partial l}{\partial \rho} = & -\frac{1}{2} tr\{\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2[\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}]\mathbf{Z}^T\} + \\ & + \frac{1}{2}(\hat{\boldsymbol{\theta}} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2[\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}]\mathbf{Z}^T\mathbf{V}^{-1})(\hat{\boldsymbol{\theta}} - \mathbf{X}\boldsymbol{\beta}) \end{aligned} \quad (34)$$

with $\mathbf{D} = (\mathbf{I} - \rho\mathbf{W})$ and $\mathbf{V} = diag(\psi_i) + \sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}$. The $\mathcal{I}(\sigma_u^2, \rho)$ matrix is

$$\mathcal{I}(\sigma_u^2, \rho) = \begin{bmatrix} \frac{1}{2} tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\} & \frac{1}{2} tr\{\mathbf{V}^{-1}\mathbf{Z}\mathbf{C}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\} \\ \frac{1}{2} tr\{\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\} & \frac{1}{2} tr\{\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\} \end{bmatrix}. \quad (35)$$

The ML estimators $\hat{\sigma}^2_{u_{ML}}$ and $\hat{\rho}_{ML}$ can be obtained iteratively using the "scoring" algorithm (20). To overcome the drawback of ML procedure, that does not consider the loss of degrees of freedom, the REML method can be used and the partial derivatives and the $\mathcal{I}_R(\sigma_u^2, \rho)$ are given by

$$\begin{aligned} s_{R_{\sigma_u^2}}(\sigma_u^2, \rho) = \frac{\partial l_R}{\partial \sigma_u^2} = & -\frac{1}{2} tr\{\mathbf{P}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\} + \frac{1}{2}\hat{\boldsymbol{\theta}}^T\mathbf{P}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{P}\hat{\boldsymbol{\theta}} \\ s_{R_\rho}(\sigma_u^2, \rho) = \frac{\partial l_R}{\partial \rho} = & -\frac{1}{2} tr\{\mathbf{P}\mathbf{Z}\sigma_u^2[\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}]\mathbf{Z}^T\} + \\ & + \frac{1}{2}\hat{\boldsymbol{\theta}}^T\mathbf{P}\mathbf{Z}\sigma_u^2[\mathbf{D}^{-1}\mathbf{W}\mathbf{C}^{-1}]\mathbf{Z}^T\mathbf{P}\hat{\boldsymbol{\theta}} \end{aligned} \quad (36)$$

$$\mathcal{I}_R(\sigma_u^2, \rho) = \begin{bmatrix} \frac{1}{2} tr\{\mathbf{P}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{P}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\} & \frac{1}{2} tr\{\mathbf{P}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{P}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\} \\ \frac{1}{2} tr\{\mathbf{P}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{P}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\} & \frac{1}{2} tr\{\mathbf{P}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{P}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\} \end{bmatrix} \quad (37)$$

with $\mathbf{P} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{V}^{-1}$ as above. Then the "scoring" algorithm (20) is used and at convergence the REML estimators are obtained.

4.2.2 MSE of Spatial EBLUP.

The MSE of the Spatial BLUP $\tilde{\theta}_i^C(\sigma_u^2, \rho)$ can be easily obtained from the general result (Saei and Chambers, 2003)

$$MSE[\tilde{\theta}_i^C(\sigma_u^2, \rho)] = g_{1i}(\sigma_u^2, \rho) + g_{2i}(\sigma_u^2, \rho) \quad (38)$$

with

$$\begin{aligned} g_{1i}(\sigma_u^2, \rho) = \mathbf{b}_i^T\{\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1} - \sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^T \times \\ \times\{diag(\psi_i) + \mathbf{Z}\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^{-1}\}^{-1}\mathbf{Z}\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\}\mathbf{b}_i \end{aligned} \quad (39)$$

and

$$\begin{aligned} g_{2i}(\sigma_u^2, \rho) = (\mathbf{x}_i - b_i^T\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^T\{diag(\psi_i) + \mathbf{Z}\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^{-1}\}^{-1}\mathbf{X}) \times \\ \times(\mathbf{X}^T\{diag(\psi_i) + \mathbf{Z}\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^{-1}\}^{-1}\mathbf{X})^{-1} \times \\ \times(\mathbf{x}_i - \mathbf{b}_i^T\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^T\{diag(\psi_i) + \mathbf{Z}\sigma_u^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}^{-1}\}^{-1}\mathbf{X})^T. \end{aligned} \quad (40)$$

The second order approximation of the Spatial EBLUP $\tilde{\theta}_i^C(\sigma_u^2, \rho)$, under normality of the errors, reduces to

$$MSE[\tilde{\theta}_i^C(\hat{\sigma}_u^2, \hat{\rho})] \approx g_{1i}(\sigma_u^2, \rho) + g_{2i}(\sigma_u^2, \rho) + g_{3i}(\sigma_u^2, \rho) \quad (41)$$

where $g_{3i}(\sigma_u^2, \rho)$ is approximately

$$\begin{aligned} tr\Bigg\{ & \begin{bmatrix} \mathbf{b}_i^T\left(\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\right) \\ \mathbf{b}_i^T\left(\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\right) \end{bmatrix} \mathbf{V} \times \\ \times & \begin{bmatrix} \mathbf{b}_i^T\left(\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\right) \\ \mathbf{b}_i^T\left(\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1} + \sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\right) \end{bmatrix}^T \bar{\mathbf{V}}(\hat{\sigma}_u^2, \hat{\rho}) \Bigg\}. \end{aligned} \quad (42)$$

In practical application σ_u^2 and ρ are substituted with asymptotically consistent estimators $\hat{\sigma}_u^2$, $\hat{\rho}$. An approximately unbiased estimator of $MSE[\tilde{\theta}_i^C(\sigma_u^2,\rho)]$, similar to (28) is given by

$$mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})] \approx g_{1i}(\hat{\sigma}_u^2,\hat{\rho}) + g_{2i}(\hat{\sigma}_u^2,\hat{\rho}) + 2g_{3i}(\hat{\sigma}_u^2,\hat{\rho}) \tag{43}$$

if $\hat{\sigma}_u^2$ and $\hat{\rho}$ are obtained by REML method. Otherwise, if $\hat{\sigma}_u^2$ and $\hat{\rho}$ are ML estimators, the $mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})]$ is given by

$$mse[\tilde{\theta}_i^C(\hat{\sigma}_u^2,\hat{\rho})] \approx g_{1i}(\hat{\sigma}_u^2,\hat{\rho}) - \mathbf{b}_{ML}^T(\hat{\sigma}_u^2,\hat{\rho}) \bigtriangledown g_{1i}(\hat{\sigma}_u^2,\hat{\rho}) + g_{2i}(\hat{\sigma}_u^2,\hat{\rho}) + 2g_{3i}(\hat{\sigma}_u^2,\hat{\rho}) \tag{44}$$

with $\bigtriangledown g_{1i}(\sigma_u^2,\rho)$ expressed as

$$\mathbf{b}_i^T \left\{ \begin{array}{c} (\mathbf{D}^{-1} - [\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1} + \sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{Z}\sigma_u^2\mathbf{D}^{-1} + \\ (\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1} - [\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1} + \sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T(-\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{Z}\sigma_u^2\mathbf{D}^{-1} + \end{array}\right.$$

$$\left. \begin{array}{c} +\sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T V^{-1}\mathbf{Z}\mathbf{D}^{-1}]) \\ +\sigma_u^2\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}]) \end{array}\right\} \mathbf{b}_i \tag{45}$$

and

$$\mathbf{b}_{ML}^T(\sigma_u^2,\rho) = \frac{1}{2m}\left\{\mathcal{I}^{-1}(\sigma_u^2,\rho)\left[\begin{array}{c} tr[(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T(-\mathbf{V}^{-1}\mathbf{Z}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{X}] \\ tr[(\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^T(-\mathbf{V}^{-1}\mathbf{Z}\sigma_u^2\mathbf{D}^{-1}\mathbf{W}\mathbf{D}^{-1}\mathbf{Z}^T\mathbf{V}^{-1})\mathbf{X}] \end{array}\right]\right\} \tag{46}$$

where $\mathcal{I}(\sigma_u^2,\rho)$ is given by (35).

5 Simulation Study

In order to asses the use of the developed methodology, simulated experiments were carried out according to the following rules:

1. the true small area means $\bar{Y}_i$ were obtained by a regression model with SAR dispersion matrix with a established spatial autoregressive coefficient (ρ) and neighborhood structures (W) obtained as follows: fixed m, the number of small areas, the value 1 is assigned to the spatial weight w_{ij} if the value drawn from an uniform distribution [0,1] is greater than 0.5, 0 otherwise. The simulations are performed with ρ equal to ± 0.75, ± 0.5, ± 0.25 and $m = 25, 50$. Combining the selected spatial autoregressive coefficient and the number of small areas, 12 spatial populations are carried out.

2. For each combination of ρ and m a synthetic population was generated assuming a normal distribution model in each small area with mean $\bar{Y}_i$ and variance drawn from a uniform distribution.

3. For each synthetic population 500 simple random samples of variable size were drawn with an only constraint: there have to be at least two sample units in each small area.

4. For each sample drawn the mean of each small area has been calculated by BLUP, EBLUP, Spatial BLUP (12), Spatial EBLUP (16) and the estimates of the parameters σ_v^2, σ_u^2 and ρ have been obtained both by ML and REML methods. Moreover the MSE of BLUP estimator ($MSE[\tilde{\theta}_i(\sigma_v^2)]$), the MSE of Spatial BLUP estimator ($MSE[\tilde{\theta}_i^S(\sigma_u^2,\rho)]$), the MSE of EBLUP estimator ($MSE[\tilde{\theta}_i(\hat{\sigma}_v^2)]$), the MSE of Spatial EBLUP estimator ($MSE[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})]$), the estimated MSE of EBLUP estimator ($mse[\tilde{\theta}_i(\hat{\sigma}_v^2)]$) and the estimated MSE of Spatial EBLUP estimator ($mse[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})]$) have been computed.

5. To asses the relative efficiencies for selected numbers of areas the ratio among the Average Estimated Mean Square Error (A.E.MSE) of Spatial EBLUP estimator and the Average Estimated Mean Square Error of EBLUP estimator has been determined (Table 1); so the ratio among the Average Mean Square Error (A.MSE) of Spatial BLUP estimator and the A.MSE of BLUP estimator and the ratio among the A.MSE of Spatial EBLUP estimator and the A.MSE of EBLUP estimator have been calculated (Table 1).

ρ	**Number of Areas**	$\frac{\mathbf{mse}[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})]}{\mathbf{mse}[\tilde{\theta}_i(\hat{\sigma}_v^2)]}$	$\frac{\mathbf{MSE}[\tilde{\theta}_i^S(\sigma_u^2,\rho)]}{\mathbf{MSE}[\tilde{\theta}_i(\sigma_v^2)]}$	$\frac{\mathbf{MSE}[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})]}{\mathbf{MSE}[\tilde{\theta}_i(\hat{\sigma}_v^2)]}$	$\frac{\mathbf{mse}[\tilde{\theta}_i^S(\hat{\sigma}_u^2,\hat{\rho})]}{\mathbf{mse}[\tilde{\theta}_i(\hat{\sigma}_v^2)]}$ **REML**
$\rho = \mathbf{-0.75}$	25	0.89	0.95	0.95	0.95
	50	0.77	0.89	0.89	0.86
$\rho = \mathbf{-0.5}$	25	1.00	0.99	0.99	1.02
	50	0.86	0.95	0.95	0.87
$\rho = \mathbf{-0.25}$	25	1.03	1.01	1.02	1.05
	50	1.02	0.99	0.99	1.03
$\rho = \mathbf{0.25}$	25	1.01	0.98	0.98	1.01
	50	0.96	0.96	0.96	0.94
$\rho = \mathbf{0.5}$	25	0.94	0.89	0.89	0.97
	50	0.88	0.83	0.83	0.90
$\rho = \mathbf{0.75}$	25	0.86	0.74	0.75	0.97
	50	0.82	0.71	0.72	0.77

Table 1: The efficiency ratio among Spatial BLUP-EBLUP estimators and BLUP-EBLUP estimators.

The columns (3) and (6) are the most interesting because they show the gain from modelling the spatial correlation among small area random effects. If the spatial autoregressive coefficient increases, weather in positive term or in negative term, the improvement in the accuracy of the estimation is high. This benefit is bigger as much as the number of small areas increase. In Figure 1 is reported the distribution of Spatial EBLUP for 16 of 25 areas; it can be note that the estimator is distributed around the true value (line blue) almost in each area.

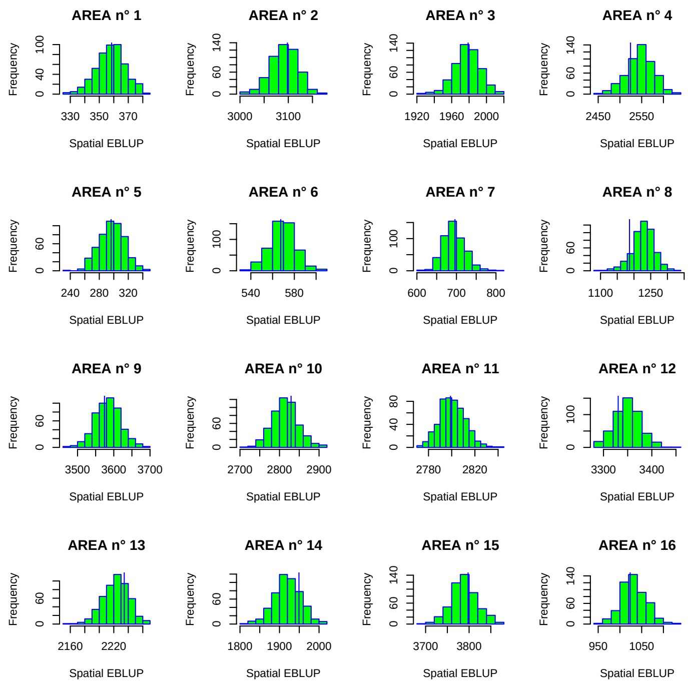

Figure 1: The distribution of the Spatial EBLUP estimator for 16 of 25 areas.

6 Final remarks

The results of this study suggest that the proposed Spatial small area estimator, which takes into account the spatial dimensions of the data modelling the spatial correlation among small area random effects, allows to obtain an appreciable improvement of the small area estimates. In practical application some of the areas are much smaller than the others and it may occur that these areas of interest are not represented in the sample. Even in this case the Spatial EBLUP can be employed: it can be used the neighborhood structure of those areas which are represented in the sample to estimate the parameters σ_u^2 and ρ. For the area i-th with sample observation the Spatial EBLUP (16) is applied; for the area i-th with no sample observation the Spatial EBLUP of θ_i can be seen as $\mathbf{x}_i\hat{\beta}$. Then the estimated parameters are employed to the complete neighborhood structure, that is, both with areas that are represented in the sample and with areas that are not represented in the sample, to measure the Mean Square Error estimator for the estimated value in each small areas. The missing data methodology could be another solution to this problem.

The Modifiable Areal Unit Problem (MAUP) is a potential source of error that can affect spatial studies which utilize aggregate data sources. Large amounts of source data require a careful choice of aggregate zones to display the spatial variation of the data in a comprehensible manner. It is this variation in acceptable areal solution that generates the term *modifiable* (Openshaw and Taylor, 1981).

The developments of Spatial EBLUP can regard either neighborhood structures which consider a function of the distance, or, if unit-specific covariates are available, the introduction of spatial dimension in the nested error unit level regression models.

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