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Intra-daily Volume Modeling
and Prediction for
Algorithmic Trading

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Abstract

The explosion of algorithmic trading has been one of the most prominent recent trends in the financial industry. Algorithmic trading consists of automated trading strategies that attempt to minimize transaction costs by optimally placing orders. The key ingredient of many of these strategies are intra-daily volume proportions forecasts. This work proposes a dynamic model for intra-daily volumes that captures salient features of the series such as time series dependence, intra-daily periodicity and volume asymmetry. Moreover, we introduce a loss functions for the evaluation of proportions forecasts which retains both an operational and information theoretic interpretation. An empirical application on a set of widely traded index ETFs shows that the proposed methodology is able to significantly outperform common forecasting methods and delivers significantly more precise predictions for VWAP trading.

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1 Introduction

Portfolio management and asset allocation require the acquisition or liquidation of positions. When the related volume is sizeable according to prevailing market conditions, placing an order is potentially able to change the price of that asset. This is particularly true for actions taken by institutional investors (e.g. pension funds or insurance companies managing large capitals) and for illiquid assets. The interaction between market participants may determine the creation of positions with the hope to profit from being on the other side of the large order. By the same token, large orders may need so-called price concessions in order to attract an adequate counterparty. The decision to buy or sell an asset in large quantities, in other words, must be informed as of the potential price impact which that particular trade may have (an effect known as *slippage*). This may result in lower profits or higher losses if the order is executed (transaction risk) or in the order not being executed at all.

In recent years, and increasingly so, services are being offered by specialized firms which provide *program trading* to institutional investors under the premise that their expertise will translate into a more efficient management of the transactions, minimizing slippage, or even into the assumption of some transaction risk. To be clear, there is no easy solution to transaction risk: the uncertainty around the actual execution price relative to one's own expected price (or of the execution of the order itself) must be weighed against the unavoidable uncertain market movements to face, should one decide to wait to place the order (*market risk*). To this extent, the relevant strategy is to plan how to place the orders relative to the characteristics of the financial market (rules and regulations, e.g. opening price formation), of the particular asset (e.g. liquidity, volatility, etc.), and, at a more advanced level, of that asset relative to other assets in a portfolio (e.g. correlation, common features, etc.).

Algorithmic trading (a.k.a. algo-trading) is widely used by investors who want to manage the market impact of exchanging large amounts of assets. It is favored by the development and diffusion of computer-based pattern recognition, so that information is processed instantaneously and action is taken accordingly with limited (if any) human judgment and intervention. The size of orders generated and executed by algo-trading is quite large and is increasing. In October 2006, the NYSE has boosted a mixed system of electronic and face-to-face auction which brings automated trades to about 50% of total trades, and similar trends are valid for other financial markets (smaller proportions when assets are more complex, e.g. options). It is generally recognized that algorithmic trading has reduced the average trade size (smaller liquidity) in the markets and hence has pushed institutional investors to split their orders in order to seek better price execution (cf. Chordia *et al.* (2008)).

The daily Volume Weighted Average Price (VWAP) was introduced by Berkowitz *et al.* (1988) as a weighted average (calculated at the end of the day) of intra-daily transaction prices with weights equal to the relative size of the corresponding traded volume to the total volume traded during the day (defined as full VWAP in Madhavan (2002)). In the original paper, the difference between the price of a trade and the recorded VWAP was used to measure the market impact of that trade. The goal of institutional investors is

to minimize such impact. VWAP is a very transparent measure, easily calculated at the end of the day with tick-by-tick data: it allows to evaluate how average traded prices were favorable to the trader. A VWAP replicating strategy is thus defined as a procedure for splitting a certain number of shares into smaller size orders during the day, which will be executed at different prices with the net result of an average price that is close to the VWAP. An interesting feature of this type of strategies is that accurate intra-daily volume proportions forecasting leads to accurate VWAP replication. Whether the VWAP benchmark is proposed on an agency base or on a guaranteed base (in exchange for a fee) is a technical aspect which does not have any bearings in what we discuss.

This paper deals with volume forecasting for VWAP trading. The trade to be executed is treated as exogenously determined (cf. Bertsimas and Lo (1998), Almgren and Chriss (2000), Engle and Ferstenberg (2007)). In order to implement the replicating strategy, we assume that we are price takers and no effort will be put in predicting prices while we concentrate on modeling volumes and predicting intra-daily volume proportions. As we will show in what follows, there are different components in the dynamics of traded volumes recorded at intra-daily intervals (relative to outstanding shares). We concentrate on single assets and we record intra-daily behavior at regular intervals. From an initial descriptive analysis of the series we derive some indications as of what features the model should reproduce. Beside the well documented U-shaped pattern of intra-daily trading activity which translates into a periodic component, we find that there are two other components which relate to a daily evolution of the volumes and to intra-daily non-periodic dynamics, respectively. We use these findings as a guideline to specify an extension of the Multiplicative Error Model (Engle (2002)) called a *Component Multiplicative Error Model* (CMEM) where each element has its own dynamic specification. The model is specified in a semiparametric fashion, thus avoiding the choice of a specific distribution of the error term. We estimate all the parameters at once by Generalized Method of Moments. The estimated model can then be used to dynamically forecast intra-daily volumes proportions. To our knowledge there is no well established methodology to evaluate proportion forecasts. In this work we introduce a loss functions, the *Slicing Loss* function, for the evaluation of proportions forecasts which retains both an operational and information theoretic interpretation.

To be sure, our approach is just the first step into the implementation of an actual VWAP based strategy. Microstructure considerations put institutional investors in a different position from those traders who exploit intra-daily volatility and are not constrained by specific choices of assets. In the interaction between the two types, the latter will scan the books to detect whether some peculiar activity may reveal the presence of a large order placed by the former. At any rate, some orders may still be too large (relative to daily volume) to be filled in one day, so that the market impact is possibly unavoidable.

Our model shares the same logic as the component GARCH model suggested by Engle *et al.* (2006b), to model intra-daily volatility. The main difference lies in the evolution of the daily and intra-daily components. Exploiting the scheme proposed here, all parameters of the model can be estimated simultaneously, instead of recurring to a multi-step procedure. Engle *et al.* (2006a) propose econometric techniques for transaction cost analysis. Some connections can be found also with P-GARCH models introduced by Boller-

slev and Ghysels (1996); relative to their suggestion, we achieve a simplification of the specification by imposing the same periodic pattern to the model coefficients (but see also Martens *et al.* (2002)). The literature on econometric models for intra-daily patterns of financial time series is quite substantial: from the initial contributions on price volume relationship (cf. the survey by Karpoff (1987)), the idea of relating intra-daily volatility and trading volumes as a function of an underlying latent information flow is contained in Andersen (1996). More recently, attention was specifically devoted to measuring the amount of liquidity of an asset based on the relationship between volume traded and price changes: Gouriéroux *et al.* (1999) concentrate on modeling weighted durations, that is the time needed to trade a given level (in quantity or value) of an asset. Dufour and Engle (2000) look at the time between trades and how that has an impact on price movements. Białkowski *et al.* (2008) concentrate on volume dynamics and take a factor analysis approach in a multivariate framework in which there is a common volume component to all stocks in an index and idiosyncratic components related to each stock which evolves according to a SETAR model. At any rate, the approach proposed here is quite general, given that some features of volumes are common to other non-negative intra-daily financial time series, such as realized volatilities, number of trades and average durations.

In this paper, we start from stylized facts (Section 2) to motivate the Component MEM (Section 3). Section 3.3 contains the details on the estimation procedure. The empirical application is divided up between model estimation and diagnostics 3.4 and volume forecasting and VWAP forecast comparisons 4. Concluding remarks follow (Section 5).

2 The Empirical Regularities of Intra-daily Volumes

We chose to analyze Exchange Traded Funds (ETFs), innovative financial products which allow straightforward trading in market averages as if they were stocks, while avoiding the possible idiosyncracies of single stocks. In the present framework, we count on a dataset consisting of regularly spaced intra-daily turnover and transaction price data for three popular equity index ETFs: DIA (Dow Jones ETF), QQQQ (Nasdaq ETF) and SPY (S&P 500 ETF). The corresponding turnover series are defined as the ratio of intra-daily transaction volume over the number of daily shares outstanding multiplied by 100. The frequency of the intra-daily data is 30 minutes, leading to 13 intra-daily bins. Volumes are computed as the sum of all transaction volumes exchanged within each intra-daily bin, while we use the last recorded transaction price before the end of each bin. The sample period used in the analysis spans from January 2002 to December 2006 and we only consider days in which there are no empty bins, which corresponds to 1248 trading days and 16224 observations. The ultra high-frequency data used in the analysis are extracted from the TAQ while shares outstanding are taken from the CRSP. Details on the series handling and management are documented in Brownlees and Gallo (2006).

We first focus on the empirical regularities of the SPY turnover series which later will be used as a guideline for the suggested model. Similar evidence also holds for the other tickers for which we report summary descriptive statistics only.

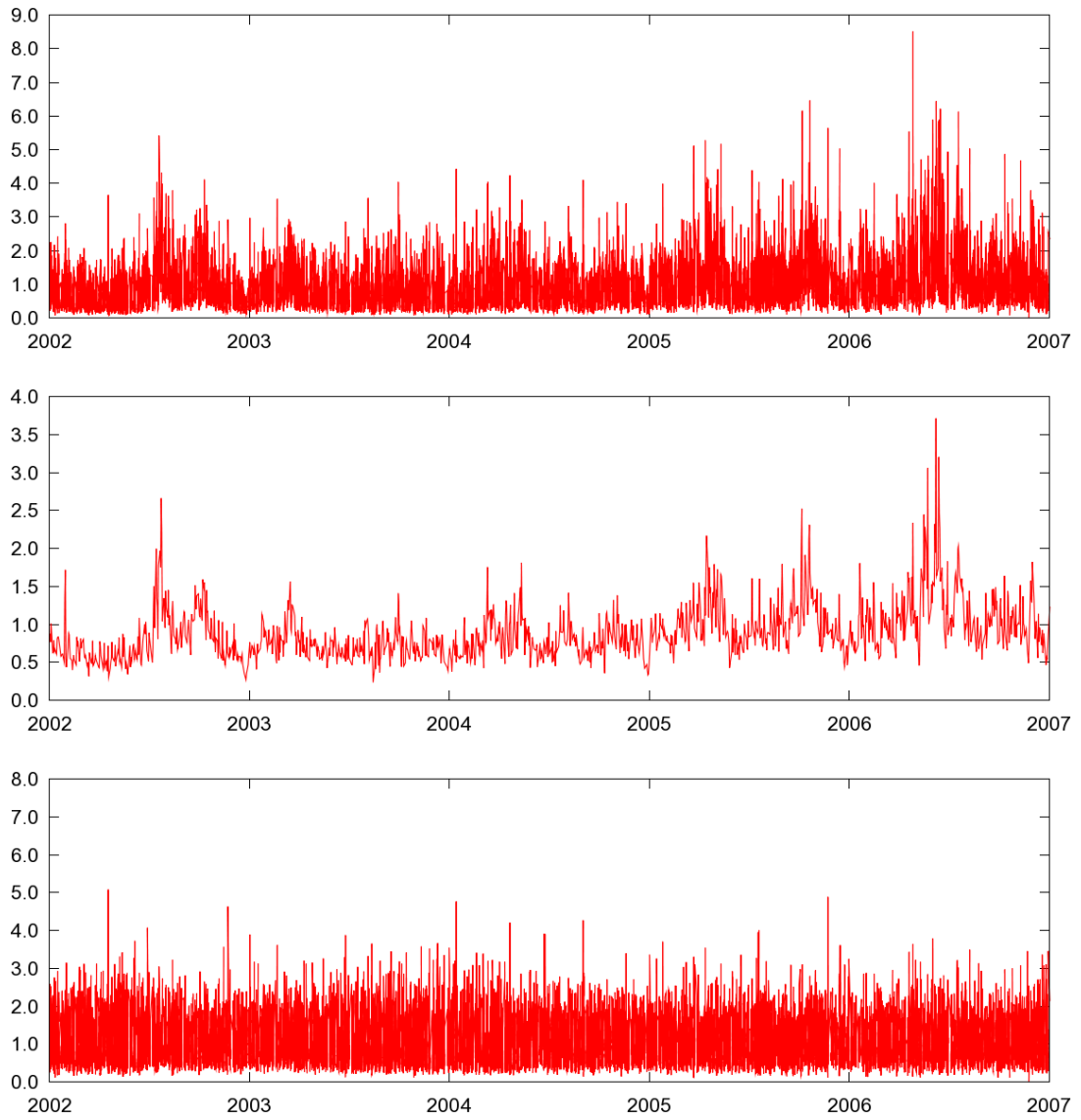


Figure 1: SPY Turnover Data: Original Turnover Data (top); Daily Averages (center), Intra-daily Component (bottom). January 2002 to December 2006.

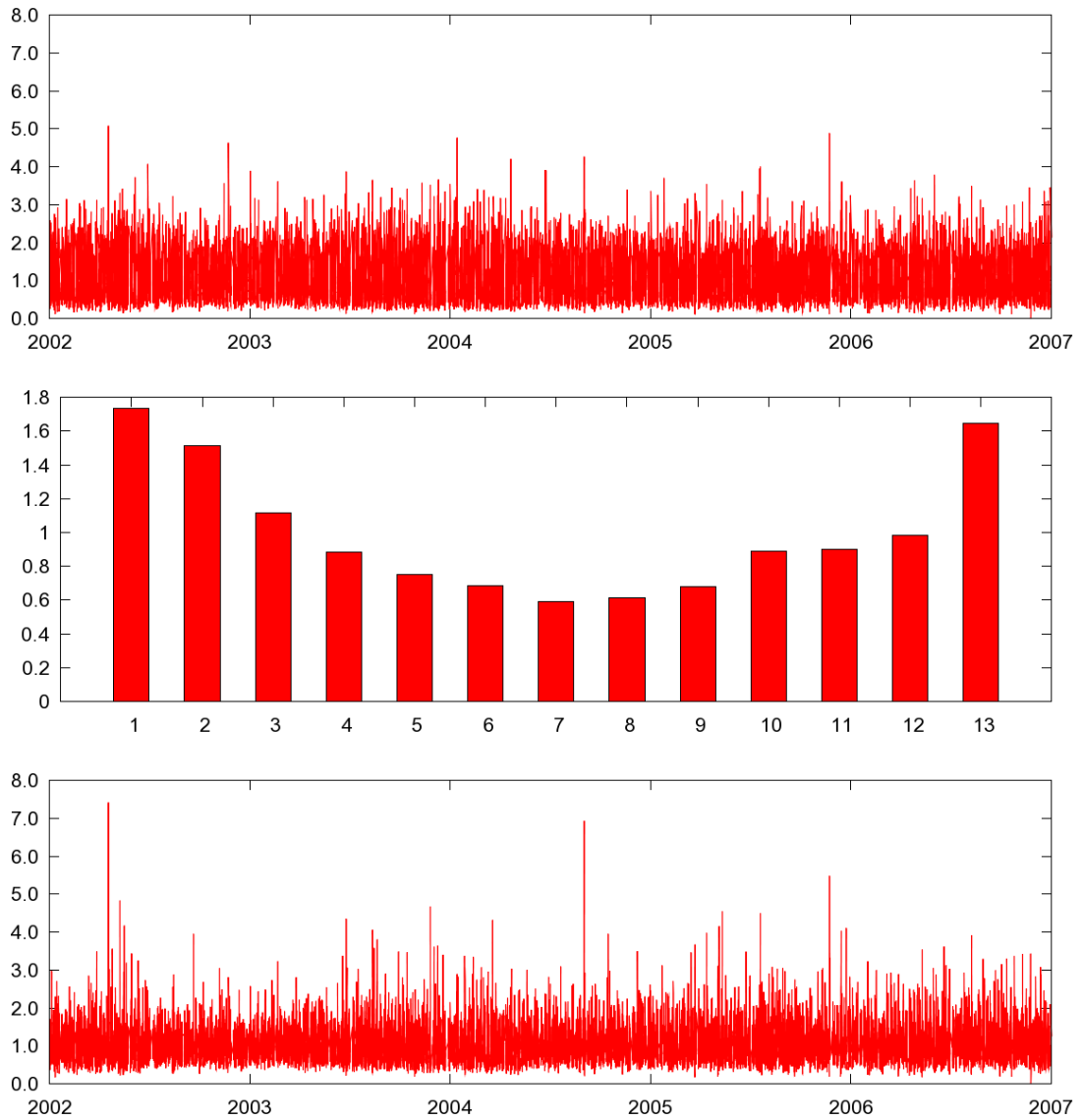


Figure 2: SPY Turnover Data: Intra-daily Component (top); Intra-daily Periodic Component (center); Intra-daily Non-periodic Component (bottom). January 2002 to December 2006.

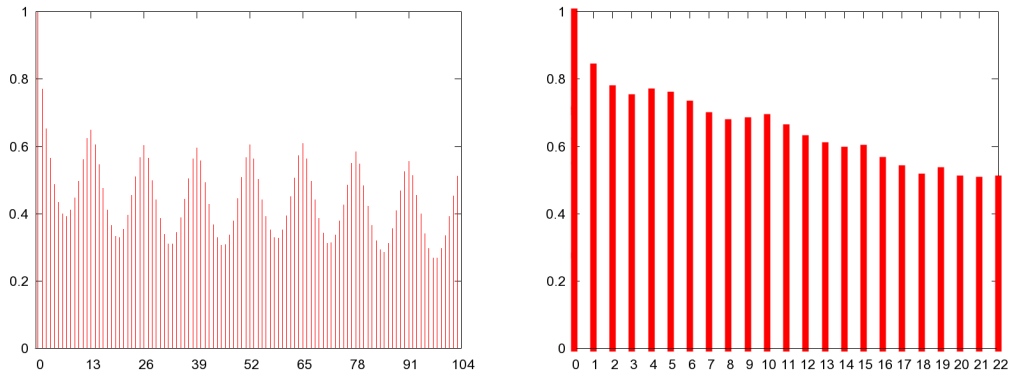


Figure 3: SPY Turnover Data: Autocorrelation Function of the Overall Turnover and of the Daily Averages.

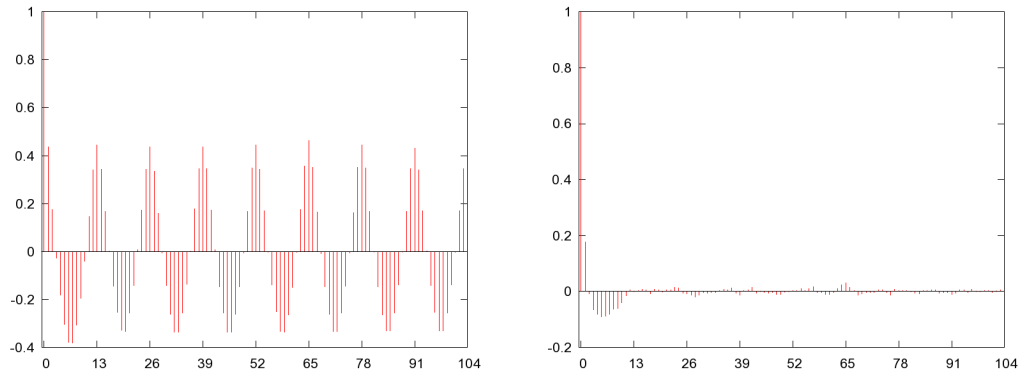


Figure 4: SPY Turnover Data: Autocorrelation Function of the Intra-daily (Periodic and Non-periodic) Components.

Let us start with a graphic appraisal of the 30–minutes turnover (top panel of Figure 1): as with most financial time series, it clearly exhibits clustering of trading activity, which is retained taking daily averages (cf. second panel of Figure 1). Dividing each observation by the corresponding daily average we obtain the intra-daily pattern (bottom panel of Figure 1); supposing to have a periodic component and a non periodic component as with other financial high–frequency data, we compute averages by time of day (13 bins – center panel of Figure 2 where the top panel reports the intra–daily series for ease of reference) which exhibit a U–shape as other intra-daily financial time series (e.g. average durations, the trading activity is higher at the opening and closing of the markets and is lower around mid-day). The ratio between these two series gives a non–periodic component which is shown in the bottom panel of Figure 2.

The dynamic features of these three series (daily, intra-daily periodic, and intra-daily non periodic) are shown in the correlograms of the original series (left panel of Figure 3) and of the daily averages (right panel). The use of unconditional intra-daily periodicity to adjust the original series results in a time series with a correlogram where periodicity is removed but some short–lived dependence is retained (Figure 4).

The autocorrelations of the components (Table 1) of the 30-minute series for the three tickers confirms the graphical analysis of SPY. The overall time series display relatively high levels of persistence which are also slowly decaying. The autocorrelations do not decrease by daily averaging. By dividing the overall turnover by its daily average (intra-daily component), a substantial part of dependence in the series is removed. Finally, once the intra-daily periodic component is removed, the resulting series show significant low order correlations only. Interestingly, the magnitudes of the various autocorrelations of the series are remarkably similar across the assets.

	overall		daily		intra-daily		intra-daily non-periodic	
	$\hat{\rho}_1$	$\hat{\rho}_{1 \text{ day}}$	$\hat{\rho}_1$	$\hat{\rho}_{1 \text{ week}}$	$\hat{\rho}_1$	$\hat{\rho}_{1 \text{ day}}$	$\hat{\rho}_1$	$\hat{\rho}_{1 \text{ day}}$
DIA	0.65	0.46	0.72	0.59	0.35	0.27	0.13	0.01
QQQQ	0.70	0.52	0.77	0.66	0.48	0.40	0.20	0.00
SPY	0.77	0.60	0.84	0.75	0.44	0.34	0.18	0.00

Table 1: Autocorrelations at selected lags of the turnover time series components. The table reports the lag 1 ($\hat{\rho}_1$) and lag 13 ($\hat{\rho}_{1 \text{ day}}$) autocorrelations of the intra-daily frequency components (overall, intra-daily and intra-daily non-periodic) and lag 1 ($\hat{\rho}_1$) and lag 5 ($\hat{\rho}_{1 \text{ week}}$) autocorrelations of the daily frequency component (daily).

3 A Multiplicative Error Model for Intra-daily Volumes

Based on the empirical regularities discussed in Section 2 we will specify the dynamics of intra-daily volumes decomposed in three components: one daily and two intra-daily (one periodic and one dynamic). Let us first establish the notation used throughout the paper. Days are denoted with $t \in \{1, \dots, T\}$; each day is divided into I equally spaced intervals (referred to as bins) indexed by $i \in \{1, \dots, I\}$. In what follows, in order to simplify the notation we may label observations indexed by the double subscript $t i$ with a single progressive subscript $\tau = I \times (t - 1) + i$. Correspondingly, we denote the total number of observations by N (equal to $T \times I$ if all I bins of data are available for all T days).

The non-negative quantity under analysis relative to bin i of day t is denoted as $x_{t i}$ or, alternatively, as x_τ . $\mathcal{F}_{t i-1}$ indicates the information about $x_{t i}$ available before forecasting it. Usually, we will assume $\mathcal{F}_{t 0} = \mathcal{F}_{t-1 I}$ but, if needed, it is possible to include additional pieces of information into $\mathcal{F}_{t 0}$, specifically related to market opening structure.

In what follows we will adopt the following convention: if $\mathbf{x}_1, \dots, \mathbf{x}_K$ are (m, n) matrices then $(\mathbf{x}_1; \dots; \mathbf{x}_K)$ represents the (mK, n) matrix obtained stacking the \mathbf{x}_t matrices columnwise.

3.1 Model Definition

Being the x_{ti} 's non-negative, a model for their daily-intra-daily dynamic can be specified by extending the logic of Multiplicative Error Models (MEM) proposed by Engle (2002). Moreover, by relying on the stylized facts showed in Section 2, we structure the model by combining different components, each one able to capture a different feature of the dynamic of the time series. We will provide further remarks about the link of the model with the empirical regularities in Section 3.2.

We then assume a Component MEM (CMEM)

$$x_{ti} = \eta_t \phi_i \mu_{ti} \varepsilon_{ti}.$$

The multiplicative innovation term ε_{ti} is assumed i.i.d., non-negative, with mean 1 and constant variance σ^2 :

$$\varepsilon_{ti} | \mathcal{F}_{ti-1} \sim (1, \sigma^2). \quad (1)$$

The conditional expectation of x_{ti} is the product of three multiplicative elements:

- η_t , a daily component;
- ϕ_i , an intra-daily periodic component aimed to reproduce the time-of-day pattern;
- μ_{ti} , an intra-daily dynamic (non-periodic) component.

In order to simplify the exposition, we assume a relatively simple specification for the components. If needed, the formulation proposed can be trivially generalized, for instance by including other predetermined variables and/or more lags (see the empirical application in Section 3.4).

The daily component is modeled as

$$\eta_t = \alpha_0^{(\eta)} + \beta_1^{(\eta)} \eta_{t-1} + \alpha_1^{(\eta)} x_{t-1}^{(\eta)} + \gamma_1^{(\eta)} x_{t-1}^{- (\eta)} \quad (2)$$

where $x^{(\eta)}$ is what we name the *deflated daily volume* and $x^{- (\eta)}$ is its ‘asymmetric’ counterpart to account for possible differences in the dynamics related to the sign of daily returns. The deflated daily volume $x^{(\eta)}$ is defined as

$$x_t^{(\eta)} = \frac{1}{I} \sum_{i=1}^I \frac{x_{ti}}{\phi_i \mu_{ti}}, \quad (3)$$

that is the daily average of the intra-daily volumes deflated by the intra-daily components ϕ_i and μ_{ti} ; and $x^{- (\eta)}$ is defined as

$$x_t^{- (\eta)} = x_t^{(\eta)} \mathbb{I}(r_t < 0)$$

with r_t denotes the daily return at day t .

The intra-daily dynamic component is specified as

$$\mu_{ti} = \alpha_0^{(\mu)} + \beta_1^{(\mu)} \mu_{ti-1} + \alpha_1^{(\mu)} x_{ti-1}^{(\mu)} + \gamma_1^{(\mu)} x_{ti-1}^{-(\mu)} \quad (4)$$

where, again, $x^{(\mu)}$ is the *deflated intra-daily volume* (and $x^{-(\mu)}$ is its ‘asymmetric’ version built on the basis of the sign of lagged bin returns). More precisely:

$$x_{ti}^{(\mu)} = \frac{x_{ti}}{\eta_t \phi_i}, \quad (5)$$

$$x_{ti}^{-(\mu)} = x_{ti}^{(\mu)} \mathbb{I}(r_{ti} < 0)$$

where r_{ti} indicates the return at bin i of day t .

Both η_t and μ_{ti} are assumed to be mean-stationary. Furthermore, μ_{ti} is constrained to have unconditional expectation equal to 1 in order to make the model identifiable. This allows us to interpret it as a pure intra-daily component and implies $\alpha_0^{(\mu)} = 1 - \beta_1^{(\mu)} - \alpha_1^{(\mu)} - \gamma_1^{(\mu)}/2$. From these assumptions we obtain also that reasonable starting conditions for the system can be $\eta_0 = x_0^{(\eta)} = \bar{x}$, $x_0^{-(\eta)} = \bar{x}/2$, $\mu_{10} = x_{1,0}^{(\mu)} = 1$ and $x_{10}^{-(\mu)} = 1/2$, where \bar{x} indicates the sample average of the modeled variable x (assuming symmetry of the returns distribution).

The intra-daily non-periodic component can be initialized with the latest quantities available, namely those computed on the previous day, i.e.

$$\mu_{t0} = \mu_{t-1I} \quad x_{t0}^{(\mu)} = x_{t-1I}^{(\mu)} \quad x_{t0}^{-(\mu)} = x_{t-1I}^{-(\mu)}$$

In synthesis, the system nests the daily and the intra-daily dynamic components by alternating the update of the former (from η_{t-1} to η_t) and of the latter (from $\mu_{t0} = \mu_{t-1I}$ to μ_{tI}). Time-varying η_t adjusts the mean level of the series, whereas the intra-daily components $\phi_i \mu_{ti}$ capture bin-specific departures from such an average level.

Note that defining $x_{ti}^{(\mu)}$ as in (5) implies $x_{ti}^{(\mu)} = \mu_{ti} \varepsilon_{ti}$. Combining this with (1) one obtains

$$E(x_{ti}^{(\mu)} | \mathcal{F}_{ti-1}) = \mu_{ti} \quad V(x_{ti}^{(\mu)} | \mathcal{F}_{ti-1}) = \mu_{ti}^2 \sigma^2 \quad (6)$$

that coincide with the properties of the corresponding quantity in the usual MEM (Engle (2002)). A similar consideration can be made for $x_t^{(\eta)}$. In fact, definition (3) implies $x_t^{(\eta)} = \eta_t \bar{\varepsilon}_t$, where $\bar{\varepsilon}_t = I^{-1} \sum_{j=1}^I \varepsilon_{tj}$, and thus

$$E(x_t^{(\eta)} | \mathcal{F}_{t-1I}) = \eta_t \quad V(x_t^{(\eta)} | \mathcal{F}_{t-1I}) = \eta_t^2 \sigma^2 / I. \quad (7)$$

On this base, $x_t^{(\eta)}$ and $x_{ti}^{(\mu)}$ are adjusted versions of the observed x_{ti} ’s that can be interpreted as carrying on an innovation contribution in the respective equations, whereas the corresponding ‘asymmetric’ versions $x^{-(\eta)}$ and $x^{-(\mu)}$ are inserted to account for possible differences in the dynamics related to the sign of daily or bin returns.

The intra-daily periodic component ϕ_i can be specified in various ways but here we retain

a parsimonious parameterization of ϕ_i via a Fourier (sine/cosine) representation:

$$\phi_i = \exp \left\{ \sum_{k=1}^K [\delta_{1k} \cos (fik) + \delta_{2k} \sin (fik)] \right\} \quad (8)$$

where $f = 2\pi/I$, $K = \lceil \frac{I+1}{2} \rceil$, $\delta_{1K} = 0$ if I is odd, $\delta_{2K} = 0$. Moreover, the number of terms into (8) may be considerably reduced if the periodic intra-daily pattern is sufficiently smooth, since few low frequencies harmonics may be enough. Alternatively, the use of shrinkage type estimation may allow to achieve flexibility and parsimony of the estimated diurnal component (cf. Brownlees and Gallo (2008)).

3.2 Discussion

3.2.1 Responce of the CMEM to the Descriptive Analysis

The daily average $\bar{x}_t = I^{-1} \sum_{i=1}^I x_{ti}$ represents a proxy of the daily component η_t . In fact, by taking its expectation conditionally on the previous day, we have

$$E(\bar{x}_t | \mathcal{F}_{t-1}) = \eta_t \frac{1}{I} \sum_{i=1}^I \phi_i E(\mu_{ti} | \mathcal{F}_{t-1}) \simeq \eta_t \frac{1}{I} \sum_{i=1}^I \phi_i = \eta_t \bar{\phi}, \quad (9)$$

where the approximate equality can be justified by noting that the non-periodic intra-daily component μ_{ti} has unit unconditional expectation, so that we can reasonably guess that it moves around this value.¹

Once the daily average is computed, the ratio $x_{ti}^{(I)} = x_{ti}/\bar{x}_t$ can be used as a proxy of the whole intra-daily component $\phi_i \mu_{ti}$, since

$$x_{ti}^{(I)} = \frac{x_{ti}}{\bar{x}_t} \simeq \frac{\eta_t \phi_i \mu_{ti} \varepsilon_{ti}}{\eta_t \bar{\phi}} = \frac{\phi_i \mu_{ti} \varepsilon_{ti}}{\bar{\phi}}. \quad (10)$$

The bin average of the quantities into (10), namely $\bar{x}_{.i}^{(I)} = T^{-1} \sum_{t=1}^T x_{ti}^{(I)}$, represents a proxy of the intra-daily periodic component ϕ_i . In fact,

$$\bar{x}_{.i}^{(I)} = \frac{1}{T} \sum_{t=1}^T x_{tj}^{(I)} \simeq \frac{\phi_i}{\bar{\phi}} \frac{1}{T} \sum_{t=1}^T \mu_{tj} \varepsilon_{tj}. \quad (11)$$

By taking its expectation conditionally on the starting information, we have

$$E(\bar{x}_{.i}^{(I)} | \mathcal{F}_{0I}) \simeq \frac{\phi_i}{\bar{\phi}} \frac{1}{T} \sum_{t=1}^T E(\mu_{ti} | \mathcal{F}_{0I}) \simeq \frac{\phi_i}{\bar{\phi}}. \quad (12)$$

¹We remark as the log formulation of the intra-daily periodic component guarantees $\prod_{i=1}^I \phi_i = 1$ but not $\bar{\phi} = 1$. However, for the applications considered $\bar{\phi}$ is quite close to one.

The last approximation can be motivated by considering that the average of the μ_{ti} 's for bin j converges, in some sense, to the unconditional average 1.

Finally, the residual quantity $x_{ti}^{(I)}/\bar{x}_{.i}^{(I)} = x_{ti}/\bar{x}_{.i}$ can be justified as proxy of the intradaily non-periodic component, since

$$\frac{x_{ti}^{(I)}}{\bar{x}_{.i}^{(I)}} \simeq \frac{\phi_i \mu_{tj} \varepsilon_{tj} / \bar{\phi}}{\phi_i / \bar{\phi}} = \mu_{ti} \varepsilon_{ti}. \quad (13)$$

3.2.2 CMEM and Component GARCH

The CMEM of Section 3.1 has some relationships with the component GARCH model suggested by Engle *et al.* (2006b), for modeling intra-daily volatility. Our proposal differs however in many points. In particular, the main difference lies in the evolution of the daily and intra-daily components. Exploiting the scheme proposed, all parameters of the model can be estimated jointly, instead of recurring to a multi-step procedure.

3.2.3 CMEM and Periodic GARCH

The structure of the CMEM shares some features with the P-GARCH model (Bollerslev and Ghysels (1996)) as well. By grouping intra-daily components ϕ_i and μ_{ti} and referring to Equation (4) for the latter, the combined component can be written as

$$\phi_i \mu_{ti} = \alpha_{0i}^{(\mu)} + \beta_{1i}^{(\mu)} \mu_{ti-1} + \alpha_{1i}^{(\mu)} x_{ti-1}^{(\mu)} + \gamma_{1i}^{(\mu)} x_{ti-1}^{-(\mu)}, \quad (14)$$

where

$$\alpha_{0i}^{(\mu)} = \alpha_0^{(\mu)} \phi_i \quad \alpha_{1i}^{(\mu)} = \alpha_1^{(\mu)} \phi_i \quad \beta_{1i}^{(\mu)} = \beta_1^{(\mu)} \phi_i \quad \gamma_{1i}^{(\mu)} = \gamma_1^{(\mu)} \phi_i. \quad (15)$$

In practice, those defined in (15) are periodic coefficients: their pattern is ruled by ϕ_i but each of them is rescaled by a (possibly) different value. The main difference relative to the P-GARCH formulation lies in the considerable simplification obtained by imposing the same periodic pattern to all coefficients. In this respect, we are inspired by the results in Martens *et al.* (2002) that a relatively parsimonious formulation, based on an intra-daily periodic component scaling the dynamical (GARCH-like) component of the variance, provides forecasts of the intra-daily volatility that are only marginally worse of a more computationally expensive P-GARCH. Martens *et al.* (2002) provide also empirical evidence in favor of the exponential formulation of the periodic intra-daily component and support its representation in a Fourier form (even if they consider only to the first 4 harmonics in their application). This notwithstanding, we depart from their approach in at least two substantial points: we include an explicit dynamic structure for the daily component, interpreting the intra-daily component as a corresponding scale factor; all parameters of the CMEM are estimated jointly.

3.3 Inference

Let us now illustrate how to obtain inferences on the model specified in Section 3.1. We group the main parameters of interest into the p -dimensional vector $\boldsymbol{\theta} = (\boldsymbol{\theta}^{(\eta)}; \boldsymbol{\theta}^{(s)}; \boldsymbol{\theta}^{(\mu)})$, where the three subvectors refer to the corresponding components of the model. Relative to these, the variance of the error term, σ^2 , represents a nuisance parameter.

Since the model is specified in a semiparametric way (see (1)), we focus our attention on the Generalized Method of Moments (GMM – Newey and McFadden (1994) and Wooldridge (1994)) as an estimation strategy not needing the specification of a density function for the innovation term.

Rather than by GMM, MEMs are often estimated by QMLE by maximizing the log-likelihood of the specification based on a Gamma distribution assumption for the innovation term (see Engle and Gallo (2006)). The first order conditions for the conditional mean parameters are in fact the same for the two estimators. However, the portion of the Gamma log-likelihood due to the Gamma dispersion parameter is not defined or overflows numerically when, respectively, zeros or inliers² are present in the data. On the other hand, our GMM approach is robust to such features which are common in these datasets, especially when dealing with a higher number of intra-daily bins or illiquid assets.

3.3.1 Efficient GMM inference

Let

$$u_\tau = \frac{x_\tau}{\eta_t \phi_i \mu_\tau} - 1, \quad (16)$$

where we simplified the notation by suppressing the reference to the dependency of u_τ on the parameters $\boldsymbol{\theta}$, on the information $\mathcal{F}_{\tau-1}$ and on the current value of the dependent variable x_τ . u_τ is a conditionally homoskedastic martingale difference, given that its conditional expectation is zero and its conditional variance is σ^2 . As a consequence, let us consider any $(M, 1)$ vector \mathbf{G}_τ depending deterministically on the information $\mathcal{F}_{\tau-1}$ and write $\mathbf{G}_\tau u_\tau \equiv \mathbf{g}_\tau$. We have

$$E(\mathbf{g}_\tau | \mathcal{F}_{\tau-1}) = \mathbf{0}, \quad \forall \tau, \Rightarrow E(\mathbf{g}_\tau) = \mathbf{0}, \quad (17)$$

by the law of iterated expectations; \mathbf{g}_τ is also a martingale difference.

Assuming that the absolute values of u_τ and $\mathbf{G}_\tau u_\tau$ have finite expectations, the uncorrelatedness of \mathbf{G}_τ and u_τ gives the former the role of instrument. \mathbf{G}_τ may depend on nuisance parameters, there including $\boldsymbol{\theta}$ also. We collect them into the vector $\boldsymbol{\psi}$ and, in order for us to concentrate on estimating $\boldsymbol{\theta}$, we assume for the moment that $\boldsymbol{\psi}$ is a known constant, postponing any further discussion about its role and how to inference it to the end of this section and to Section 3.3.2.

If $M = p$, we have as many equations as the dimension of $\boldsymbol{\theta}$, thus leading to the moment

²Inliers are observations that are anomalously by being too small (in this context, too close to zero).

criterion

$$\bar{\mathbf{g}} = \frac{1}{N} \sum_{\tau=1}^N \mathbf{g}_\tau = \mathbf{0}. \quad (18)$$

Under correct specification of the η_t , ϕ_i , and μ_{ti} equations and some regularity conditions, the GMM estimator $\hat{\boldsymbol{\theta}}_N$, obtained solving (18) for $\boldsymbol{\theta}$, is consistent (Wooldridge (1994, th. 7.1)). Furthermore, under some additional regularity conditions, we have asymptotic normality of $\hat{\boldsymbol{\theta}}_N$, with asymptotic covariance matrix (Wooldridge (1994, th. 7.2))

$$\text{Avar}(\hat{\boldsymbol{\theta}}_N) = \frac{1}{N} (\mathbf{S}' \mathbf{V}^{-1} \mathbf{S})^{-1}, \quad (19)$$

where

$$\mathbf{S} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\tau=1}^N E(\nabla_{\boldsymbol{\theta}'} \mathbf{g}_\tau) \quad (20)$$

$$\mathbf{V} = \lim_{N \rightarrow \infty} \frac{1}{N} V \left(\sum_{\tau=1}^N \mathbf{g}_\tau \right) = \lim_{N \rightarrow \infty} \left[\frac{1}{N} \sum_{\tau=1}^N E(\mathbf{g}_\tau \mathbf{g}_\tau') \right]. \quad (21)$$

The last expression for V comes from the fact that \mathbf{g}_τ is a martingale difference, since this is a sufficient condition for making these terms to be serially uncorrelated; moreover, the same condition leads to simplifications in the assumptions needed for the asymptotic normality, by virtue of the martingale CLT.

The martingale difference structure of u_τ gives also a simple formulation for the *efficient* choice of the instrument \mathbf{G}_τ , where efficient is meant producing the 'smallest' asymptotic variance among the GMM estimators arisen by $\bar{\mathbf{g}}$ functions structured as in (18), with $\mathbf{g}_\tau = \mathbf{G}_\tau u_\tau$ and \mathbf{G}_τ being an instrument. Such efficient choice is

$$\mathbf{G}_\tau^* = -E(\nabla_{\boldsymbol{\theta}'} u_\tau | \mathcal{F}_{\tau-1}) V(u_\tau | \mathcal{F}_{\tau-1})^{-1}. \quad (22)$$

Computing $E(\mathbf{g}_\tau \mathbf{g}_\tau')$ into (21) and $E(\nabla_{\boldsymbol{\theta}'} \mathbf{g}_\tau)$ into (20) we obtain

$$E(\mathbf{g}_\tau \mathbf{g}_\tau') = -E(\nabla_{\boldsymbol{\theta}'} \mathbf{g}_\tau) = \sigma^2 E(\mathbf{G}_\tau^* \mathbf{G}_\tau^{*'}),$$

so that

$$\mathbf{V} = -\mathbf{S} = \sigma^2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\tau=1}^N E(\mathbf{G}_\tau^* \mathbf{G}_\tau^{*'})$$

and (19) specializes as

$$\text{Avar}(\hat{\boldsymbol{\theta}}_N) = \frac{1}{N} (\mathbf{S}' \mathbf{V}^{-1} \mathbf{S})^{-1} = -\frac{1}{N} \mathbf{S}^{-1} = \frac{1}{N} \mathbf{V}^{-1}. \quad (23)$$

Considering the analytical structure of u_τ in the model (equation (16)), we have

$$\nabla_{\boldsymbol{\theta}'} u_\tau = -\mathbf{a}_\tau(u_\tau + 1),$$

where

$$\mathbf{a}_\tau = \eta_t^{-1} \nabla_{\boldsymbol{\theta}} \eta_t + \mu_\tau^{-1} \nabla_{\boldsymbol{\theta}} \mu_\tau + \phi_i^{-1} \nabla_{\boldsymbol{\theta}} \phi_i \quad (24)$$

so that (22) becomes

$$\mathbf{G}_\tau^* = \mathbf{a}_\tau \sigma^{-2}.$$

Replacing it into $\mathbf{g}_\tau = \mathbf{G}_\tau u_\tau$ and this, in turn, into (18), we obtain that the GMM estimator of $\boldsymbol{\theta}$ in the CMEM solves the MM equation

$$\frac{1}{N} \sum_{\tau=1}^N \mathbf{a}_\tau u_\tau = \mathbf{0}, \quad (25)$$

which does not depend on the nuisance parameter σ^2 and, therefore, inference relative to the main parameter $\boldsymbol{\theta}$ does not depend on the estimation of σ^2 .

The asymptotic variance matrix of $\widehat{\boldsymbol{\theta}}_N$ is

$$\text{Avar}(\widehat{\boldsymbol{\theta}}_N) = \frac{\sigma^2}{N} \left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\tau=1}^N E(\mathbf{a}_\tau \mathbf{a}'_\tau) \right]^{-1} \quad (26)$$

that can be consistently estimated by

$$\widehat{\text{Avar}}(\widehat{\boldsymbol{\theta}}_N) = \widehat{\sigma}_N^2 \left[\sum_{\tau=1}^N \mathbf{a}_\tau \mathbf{a}'_\tau \right]^{-1} \quad (27)$$

where $\widehat{\sigma}_N^2$ is a consistent estimator of σ^2 (Section 3.3.2) and \mathbf{a}_τ is here evaluated at $\widehat{\boldsymbol{\theta}}_N$.

3.3.2 Inference on σ^2

The second moment of u_τ into (16) suggests that a natural estimator for the nuisance parameter σ^2 can be

$$\widehat{\sigma}_N^2 = \frac{1}{N} \sum_{\tau=1}^N u_\tau^2 \quad (28)$$

where u_τ denotes here the working residual (16) computed by using current values of $\widehat{\boldsymbol{\theta}}_N$. An interesting characteristic of such estimator, is that it is not compromised by zeros in the data.

3.4 Empirical Application: In Sample Volume Analysis

The empirical application focuses on the analysis of the tickers DIA, QQQQ and SPY in 2002–2006. We consider four variants of the CMEM introduced in Section 3.1 (Equations (2) and (4)):

base: CMEM with lag-1 dependence and no asymmetric effects;

asym: base CMEM with lag-1 asymmetric effects (daily and the intra-daily);

intra2: base CMEM with intra-daily autoregressive components of order 2;

asym-intra2: intra2 CMEM with lag-1 asymmetric effects (daily and intra-daily).

The parameter estimates of the daily and intra-daily components are reported in Table 2, together with residual diagnostics. The periodic component, omitted from the table, is expressed in Fourier form (Equation (8)). Also, $\alpha_0^{(\mu)}$ lacks a t-statistic because estimated via expectation targeting by imposing $E(\mu_\tau) = 1$.

Some comments are in order. The parameter estimates of each model are similar across assets, suggesting common behavior in the volume dynamics. We have a high (close to 1) level of daily persistence (measured as $\alpha^{(n)} + \beta^{(n)}$ in the symmetric, respectively, $\alpha^{(n)} + \gamma^{(n)}/2 + \beta^{(n)}$ asymmetric specifications). Contrary to customary values in a typical GARCH(1,1) estimates on daily returns, in the present context $\alpha^{(n)}$ is much larger. Intra-daily asymmetric effects are always strongly significant, while daily asymmetric effects are significant for the DIA and SPY tickers only. Their signs are always positive, coherently with the notion that negative past returns have a greater impact on the level of market activity in comparison to the positive ones. The second order intra-daily lag is negative and with a relatively large magnitude, but it is such that the Nelson and Cao (1992) non-negativity condition for the corresponding component is satisfied in all cases, and has the effect of increasing the level of the intra-daily persistence, as can be observed from the column labeled $\text{pers}(\mu)$ in table 2. Correspondingly, the less-than-satisfactory performance of serial correlation residual diagnostics (reported in the last columns of table 2) – even with asymmetric effects – is improved when the second order term is included in the dynamic intra-daily component.

4 Intra-daily Volume Forecasting for VWAP Trading

VWAP trading has become one of the most well established automated trading models other recent years. A discussion on these type of trading procedures can be found, for instance, in Madhavan (2002).

A VWAP trading strategy is defined as a procedure for splitting a certain number of shares into smaller size orders during the day in the attempt to obtain an average execution price that is close to the daily VWAP. Let the VWAP for day t be defined as

$$\text{VWAP}_t = \frac{\sum_{j=1}^{J_t} v_t(j) p_t(j)}{\sum_{j=1}^{J_t} v_t(j)}.$$

where $p_t(j)$ and $v_t(j)$ denote respectively the price and volume of the j -th transaction of day t and J_t is the total number of trades of day t . For a given partition of the trading day

into I bins, it is possible to express the numerator of the VWAP as

$$\begin{aligned}\sum_{j=1}^{J_t} v_t(j) p_t(j) &= \sum_{i=1}^I \left(\sum_{j \in \mathcal{J}_i} v_t(j) \right) \bar{p}_{ti} \\ &= \sum_{i=1}^I x_{ti} \bar{p}_{ti},\end{aligned}$$

where \bar{p}_{ti} is the VWAP of the i -th bin and \mathcal{J}_i denotes the set of indices of the trades belonging to the i -th bin. Hence,

$$\text{VWAP}_t = \frac{\sum_{i=1}^I x_{ti} \bar{p}_{ti}}{\sum_{i=1}^I x_{ti}} = \sum_{i=1}^I w_{ti} \bar{p}_{ti} = \mathbf{w}'_t \bar{\mathbf{p}}_t$$

where w_{ti} is the intra-daily proportion of volumes traded in bin j on day t , that is $w_{ti} = x_{ti} / \sum_{i=1}^I x_{ti}$. Let $\mathbf{y} = (y_1, \dots, y_I)$, an order slicing strategy over day t with the same bin intervals. We can define the Average Execution Price as the quantity

$$\text{AEP}_t = \sum_{i=1}^I y_i \bar{p}_{ti} = \mathbf{y}' \bar{\mathbf{p}}_t,$$

where the assumption is made that the traders are price takers and execute at or *close* to the average price (more on this later). The choice variable being the vector \mathbf{y} , we can solve the problem of minimizing the distance between the two outcomes in a mean square error sense, namely

$$\min_{\mathbf{y}} \delta_t = (\mathbf{w}'_t \bar{\mathbf{p}}_t - \mathbf{y}' \bar{\mathbf{p}}_t)^2,$$

where, solving the minimization problem leads to the first order conditions

$$\frac{d\delta_t}{d\mathbf{y}} = \mathbf{0} \Rightarrow -2\bar{\mathbf{p}}_t (\mathbf{w}_t - \mathbf{y})' \bar{\mathbf{p}}_t = \mathbf{0}, \quad (29)$$

which has a meaningful solution for $\mathbf{y} = \mathbf{w}_t$, that is when the order slicing sequence for each sub period in the day reproduces exactly the overall relative volume for that sub period.

The implication of Equation (29) is that the VWAP replication problem can be cast as an intra-daily volume proportion forecasting problem: the better we can predict the intra-daily volumes proportions, the better we can track VWAP.

4.1 VWAP Replication Strategies

Following Białkowski *et al.* (2008), we consider two types of VWAP replication strategies: Static and Dynamic. The Static VWAP replication strategy assumes that the order slicing is set before the market opening and it is not revised during the trading day. In the Dynamic VWAP replication strategy scenario on the other hand, order slicing is revised

at each new sub period as new intra-daily volumes are observed.

Let $\widehat{x}_{t i|t-1}$ be shorthand notation for the prediction of $x_{t i}$ conditionally on the previous day full information set $\mathcal{F}_{t-1 I}$. The *Static VWAP* replication strategy is implemented using slices with weights given by

$$\widehat{w}_{t i|t-1} = \frac{\widehat{x}_{t i|t-1}}{\sum_{i=1}^I \widehat{x}_{t i|t-1}} \quad i = 1, \dots, I,$$

that is the proportion of volumes for bin i is given by predicted volume in bin i divided by the sum of the volume predictions.

Let $\widehat{x}_{t i|i-1}$ be shorthand notation to denote the prediction of $x_{t i}$ conditionally on $\mathcal{F}_{t i-1}$. The *Dynamic VWAP* replication strategy is implemented using slices with weights given by

$$\widehat{w}_{t i|i-1} = \begin{cases} \frac{\widehat{x}_{t i|i-1}}{\sum_{j=1}^I \widehat{x}_{t j|i-1}} \left(1 - \sum_{j=1}^{i-1} \widehat{w}_{t j|j-1}\right) & i = 1, \dots, I-1 \\ \left(1 - \sum_{i=1}^{I-1} \widehat{w}_{t i|i-1}\right) & i = I \end{cases}$$

that is, for each intra-daily bin from 1 to $I-1$ the predicted proportion is given by the proportion of 1-step ahead volumes with respect to the sum of the remaining predicted volumes multiplied by the slice proportion left to be traded. On the last period of the day I , the predicted proportion is equal to the remaining part of the slice that needs to be traded.

4.2 Forecast Evaluation

We evaluate out-of-sample performance from different perspectives: intra-daily volumes, intra-daily volume proportions and daily VWAP prediction.

A natural way to assess volume predictive ability is to consider the mean square prediction error of the volume forecasts, defined as

$$\text{MSE}^{\text{vol}} = \sum_{t=1}^T \sum_{i=1}^I (x_{t i} - \widehat{x}_{t i|\cdot})^2,$$

where $\widehat{x}_{t i|\cdot}$ denotes the volume from some VWAP replication and volume forecasting strategy. Although such a metric provides insights as to which model provides a more realistic description of volume dynamics, it does not necessarily provide useful information as to the performance of the models for VWAP trading.

Successful VWAP replication lies in predicting intra-daily volume proportions accurately. Proportions have quite different properties in comparison to a continuous variable and, to our knowledge, there are no well established loss functions in the literature for the evaluation of proportion forecasts. The ‘‘Slicing’’ loss function we propose to measure

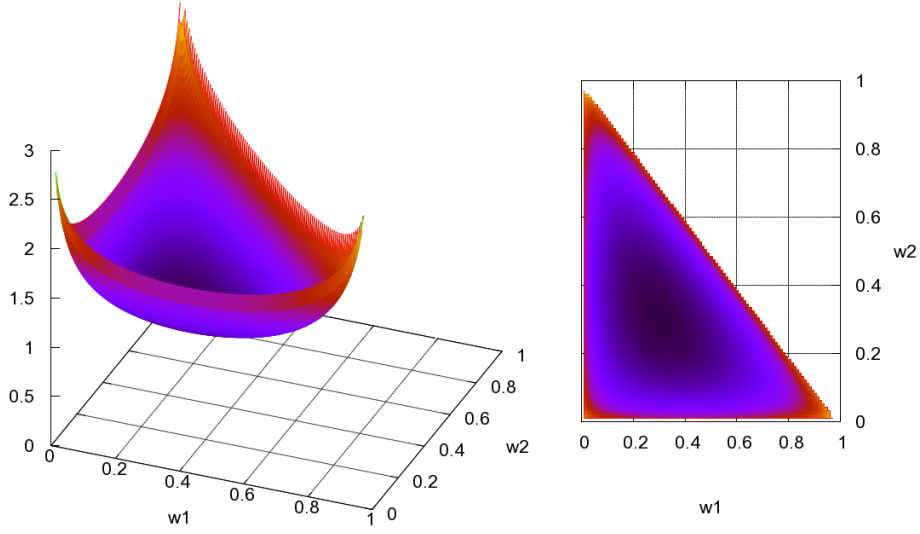


Figure 5: Slicing Loss function for $I = 3$ and $(w_1, w_2, w_3) = (0.3, 0.3, 0.4)$.

intra-daily volume proportion predictions ability is

$$L^{\text{slicing}} = - \sum_{t=1}^T \sum_{i=1}^I w_{ti} \log \hat{w}_{ti}, \quad (30)$$

which we motivate from both an “operational” as well as an information theoretic perspectives. In the spirit of Christoffersen (1998) in the context of Value at Risk, we ask ourselves which properties proportion forecast ought to have under correct specification. Assume that a broker is interested in trading n shares of the asset³ each day. If the intra-daily volume proportions predictions are correct, then the observed intra-daily volumes nw_{ti} , $i = 1, \dots, I$ behave like a sample from a multinomial distribution with parameters \hat{w}_{ti} , $i = 1, \dots, I$, and n ; that is

$$(nw_{t1}, \dots, nw_{tI}) \sim \text{Mult}(\hat{w}_{t1}, \dots, \hat{w}_{tI}, n).$$

This suggest that an appropriate loss function for the evaluation of such forecasts is the negative of the multinomial predictive log-likelihood

$$L^{\text{Mult}} = - \sum_{t=1}^T \left(\log \frac{n!}{(nw_{t1})! \dots (nw_{tI})!} + \sum_{i=1}^I nw_{ti} \log \hat{w}_{ti} \right).$$

An alternative evaluation strategy consists of computing the discrepancy between the ac-

³We are implicitly assuming for simplicity’s sake, that the actual intra-daily proportions nw_{ti} are all integer.

tual and predicted intra-daily volume proportions as the discrepancy between two discrete distributions. Using Kullback–Leibler discrepancy we get

$$L^{\text{KL}} = \sum_{t=1}^T \left(\sum_{i=1}^I w_{ti} \log w_{ti} - w_{ti} \log \widehat{w}_{ti} \right).$$

Interestingly, both the Multinomial and Kullback–Leibler losses provide equivalent rankings among competing forecasting methods in that the comparison is driven by the common term $-w_{ti} \log \widehat{w}_{ti}$. Figure 5 shows a picture of the slice loss function in the case of 3 intra-daily bins when the actual proportions w_t are (0.3, 0.3, 0.4). The slicing loss function is defined over the $I - 1$ dimensional simplex described by $\sum_{i=1}^I \widehat{w}_{ti} = 1$, has a minimum in correspondence to the true values, the value of the loss function goes to infinity on the boundaries of the simplex (when the actual proportions are in the interior of the simplex) and is evidently asymmetric.

Finally, we also consider *VWAP tracking errors* MSE as in Białkowski *et al.* (2008) defined as

$$\text{MSE}^{\text{VWAP}} = \sum_{t=1}^T \left(\frac{\text{VWAP}_t - \widehat{\text{VWAP}}_t}{\text{VWAP}_t} 100 \right)^2,$$

where VWAP_t is the VWAP of day t and $\widehat{\text{VWAP}}_t$ is the realized average execution price obtained using some VWAP replication strategy and volume forecasting method. Both VWAP_t and $\widehat{\text{VWAP}}_t$ are computed using the last recorded price of the i -th bin as a proxy of the average price of the same interval. The VWAP tracking error for day t can be seen as an average of slicing errors within each bin weighted by the relative deviation of the price associated to that bin with respect to the VWAP:

$$\text{MSE}^{\text{VWAP}} = \sum_{t=1}^T \left(\sum_{i=1}^I (w_{ti} - \widehat{w}_{ti}) \frac{\bar{p}_{ti}}{\text{VWAP}_t} \right)^2 100^2$$

Note that the deviations of the prices from the daily VWAP add an extra source of noise which can spoil the correct ranking of slice forecasts. In light of this, we recommend evaluating the precision of the forecasts by means of the Slicing loss function.

4.3 Empirical Application: Out-of-Sample VWAP Prediction

Our empirical application consists of VWAP tracking exercise of the tickers DIA, QQQQ and SPY between January 2005 and December 2006 (502 days, 6526 observations). We track VWAP using turnover predictions from our CMEM specifications using both Static and Dynamic VWAP replication strategies based on parameter estimates over the 2002–2004 data. In order to assess the usefulness of the proposed approach with respect of a simple benchmark we also track VWAP using (periodic) Rolling Means (RM), that is the predicted volume for the i -th bin is obtained as the mean over the last 40 days at the

same bin. The Rolling Means are used to track VWAP using the Static VWAP replication approach.

Table 3 reports the volume MSE, slicing loss and VWAP tracking MSE together with asterisks denoting the significance of a Diebold-Mariano test of equal predictive ability with respect to RM using the corresponding loss functions. In term of volume and volume proportions predictions, the CMEM Dynamic VWAP Tracking performs best and significantly outperforms the benchmark, followed by the CMEM Static VWAP which generally outperforms the benchmark as well. The ranking of the CMEM specifications reflects the in-sample estimation results with models with richer intra-daily dynamics and asymmetric terms performing better. The VWAP tracking MSE delivers mixed evidence. While it is true that the CMEM Dynamic VWAP replication strategy achieves the best out-of-sample performance, statistical significance is less clear cut. It is strong for DIA (and extends to the Static model), but it is less so for QQQQ and SPY. However, as mentioned in the previous section, the VWAP tracking MSE does not necessarily provide a good assesment of volume proportion forecasts which should be based on the slicing loss function.

5 Conclusions

In this paper we suggested a dynamic model with different components which captures the behavior of traded volumes (relative to outstanding shares) viewed from daily and (periodic and non-periodic) intra-daily time perspectives. The ensuing Component Multiplicative Error Model is well suited to be simultaneously estimated by Generalized Method of Moments. The application to three major ETFs shows that both the static and the dynamic VWAP replication strategies generally outperform a naïve method of rolling means for intra-daily volumes.

We would need to extend the analysis to a wider group of tickers to check whether the stylized facts are shared by other classes of assets (e.g. single stocks) and to investigate whether overall market capitalization or the percentage of holdings by institutional investors have a bearing on the characteristics of the estimated dynamics.

The CMEM can be used in other contexts in which intra-daily bins are informative of some periodic features (e.g. volatility, number of trades, average durations) together with an overall dynamic which has components at different frequencies. The periodic component can be more parsimoniously specified by recurring to some shrinkage estimation as in Brownlees and Gallo (2008). Multivariate extensions are also possible (following Cipollini *et al.* (2008) by retrieving the price-volume dynamics mentioned earlier in order to establish a relationship that can be related to the flow of information at different frequencies, separating it from (possibly common) periodic components.

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Ticker	Specification	$\alpha_0^{(\eta)}$	$\alpha_1^{(\eta)}$	$\gamma^{(\eta)}$	$\beta^{(\eta)}$	$\alpha_0^{(\mu)}$	$\alpha_1^{(\mu)}$	$\alpha_2^{(\mu)}$	$\gamma^{(\mu)}$	$\beta^{(\mu)}$	σ	pers(μ)	LB_1	LB_7	LB_{13}
DIA	base	0.025 4.388	0.320 11.835		0.654 22.287	0.268	0.316 33.73			0.416 23.813	0.674	0.732	0.122	0.000	0.000
	asym	0.023 4.302	0.298 10.814	0.025 2.428	0.666 22.945	0.257	0.275 27.513		0.058 6.651	0.439 25.658	0.673	0.743	0.041	0.000	0.000
	intra2	0.011 2.963	0.180 7.154		0.808 29.663	0.060	0.347 35.181	-0.225 -16.257		0.817 51.039	0.671	0.919	0.159	0.839	0.802
	asym-intra2	0.010 2.957	0.158 6.561	0.026 3.049	0.819 32.175	0.057	0.319 30.864	-0.226 -17.643	0.042 7.966	0.830 60.655	0.668	0.926	0.181	0.802	0.805
QQQQ	base	0.019 3.741	0.363 13.156		0.622 21.716	0.286	0.377 42.346			0.337 21.157	0.483	0.714	0.057	0.000	0.000
	asym	0.019 3.772	0.348 12.327	0.014 1.699	0.63 21.985	0.281	0.350 37.285		0.043 6.416	0.348 22.115	0.481	0.674	0.037	0.000	0.000
	intra2	0.001 0.817	0.089 4.656		0.91 45.851	0.031	0.399 44.393	-0.294 -24.835		0.864 86.833	0.481	0.955	0.463	0.236	0.516
	asym-intra2	0.001 0.812	0.090 4.629	0.006 0.886	0.907 45.243	0.031	0.381 41.268	-0.291 -25.228	0.024 6.962	0.866 91.154	0.479	0.954	0.527	0.239	0.518
SPY	base	0.021 4.368	0.410 13.997		0.569 18.489	0.289	0.360 39.476			0.352 21.203	0.533	0.712	0.032	0.000	0.000
	asym	0.022 4.677	0.384 12.931	0.030 3.158	0.579 18.956	0.281	0.321 33.311		0.055 7.520	0.370 22.641	0.532	0.719	0.013	0.000	0.000
	intra2	0.009 2.718	0.213 7.518		0.778 25.828	0.044	0.384 41.261	-0.273 -21.137		0.845 64.616	0.532	0.936	0.631	0.424	0.516
	asym-intra2	0.008 2.846	0.164 6.251	0.023 2.686	0.816 29.999	0.035	0.359 37.233	-0.279 -23.907	0.032 8.576	0.870 88.53	0.531	0.952	0.705	0.237	0.238

Table 2: Parameter Estimates. Sample period 2002 – 2006 (1248 trading days, 13 daily bins, 16224 observations). t -statistics are reported in parenthesis. LB_l denote p-values of the corresponding Ljung-Box statistics at the l -th lag. pers(μ) indicates estimated persistence of the dynamic intra-daily component.

	DIA			QQQQ			SPY		
	volume	slicing	vwap	volume	slicing	vwap	volume	slicing	vwap
RM	56.68	2.5163	1.373	64.83	2.4988	1.796	52.28	2.5139	1.129
					Static				
base	53.00	2.5121	1.354	63.73	2.4987	1.822	50.45	2.5131	1.110
	***	***	**	***	***		***	***	
asym	53.04	2.5121	1.352	63.71	2.4987	1.823	50.45	2.5132	1.110
	***	***	**	***	***		***	***	
intra2	53.00	2.5108	1.335	63.73	2.4973	1.803	50.45	2.5127	1.124
	***	***	***	***	***		***	***	
asym-intra2	53.04	2.5099	1.319	63.71	2.4968	1.771	50.45	2.5123	1.120
	***	***	***	***	***		***	***	
					Dynamic				
base	49.17	2.5091	1.240	58.41	2.4962	1.753	45.39	2.5102	1.079
	***	***	***	***	***		***	***	**
asym	49.11	2.5090	1.238	58.37	2.4962	1.755	45.31	2.5101	1.082
	***	***	***	***	***		***	***	**
intra2	49.07	2.5067	1.241	57.89	2.4941	1.744	45.16	2.5089	1.095
	***	***	***	***	***		***	***	**
asym-intra2	48.91	2.5062	1.233	58.03	2.4935	1.687	44.98	2.5086	1.093
	***	***	***	***	***	*	***	***	**

Table 3: Out-of-Sample Volume, Slicing and VWAP tracking forecasting results. For each ticker, specification and VWAP replication strategy the table reports the values of the Volume, Slicing and VWAP tracking error loss functions. Asterisks denote the significance (* 1%, ** 5% and *** 10%) of a Diebold-Mariano test of equal predictive ability with respect to RM using the corresponding loss functions.

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